# PROBABILITIES OF CERTAIN ELECTION RESULTS

KENNETH MAY, Carleton College*

**1. Introduction.** In any indirect electoral system or in the election of any representative body, the results will not in general correspond exactly to the votes of the electorate. In particular, it is possible for a candidate or a party to win a majority of the electors or representatives with a minority of the popular vote. What is the probability of such a result?† The answer depends, of course, upon many factors, including the method of election, the number of districts, voters and parties, and the frequency distributions of election outcomes in the districts.

In this paper we confine ourselves to certain simple cases in order to suggest a tentative answer to the above question. We assume $n$ districts, $m$ voters per district, and two parties. In each district all voters cast votes for one or the other party according to uniform, independent distributions. Ties are excluded by taking $m$ and $n$ odd. We are interested in $P(m, n)$, the probability that the party which wins the majority of the districts gains only a minority of the total vote $mn$. Expressions are found for $P(m, n)$, $P(m, 3)$, $P(3, n)$, $P(\infty, n)$, and $P(m, \infty)$. The results are summarized in the last section.

**2. Preliminary results.** Let $x_j$ be the vote of one of the parties in the $j$th district. The vote of the other party is $m-x_j$. Since the problem is symmetric with respect to the two parties, we need to consider only the $x_j$. We assume that $x_j$ takes integral values $0 \leqq x_j \leqq m$, with equal probabilities $1/(m+1)$. The smallest $x_j$ to win a district is $\mu=(m+1)/2$. The party's total vote is $X=\sum_1^n x_j$.

Let $y_j$ be defined as follows: $y_j=0$ when $x_j<\mu$ and $y_j=1$ when $x_j \geqq \mu$. Then the number of districts won is $Y=\sum_1^n y_j$, and the minimum number of districts for a majority of the districts is $\nu=(n+1)/2$. The greatest $X$ for which $X<mn/2$ is given by $\overline{X}=(mn-1)/2=2\mu\nu-\mu-\nu$. On the other hand, in order to win a majority of the districts it is necessary to have at least $\mu$ votes in at least $\nu$ districts. Hence the minimum total vote for which it is possible to win a majority of the districts is given by $\underline{X}=\mu\nu$. Thus the probability that the party wins a majority of the districts with a minority of the total vote is the probability of the simultaneous occurrence of $\underline{X} \leqq X \leqq \overline{X}$ and $Y \geqq \nu$. Since the other party has an equal chance of winning in the same way, we have $P(m, n)=2P(\underline{X} \leqq X \leqq \overline{X}, Y \geqq \nu)$.

The problem may be restated in classical terms as follows: Given $n$ urns, in each of which are $m+1$ balls marked $0, 1, 2, \cdots, m-1, m$; when a ball is drawn from each, what is the probability that the sum of the numbers drawn is less than $mn/2$ (or greater than $mn/2$) while at the same time the number of

* Presented in part to the Minnesota Section of the Mathematical Association of America at St. Cloud, Minnesota, May 10, 1947.

† The question was proposed to the author by Professor G. L. Field of the Department of Government, Wayne University. The writer is indebted to Professors J. Neyman and H. Lewy of the University of California for helpful suggestions.

balls drawn with markings greater than $m/2$ is greater than $n/2$ (or less than $n/2$). The problem is also equivalent to finding the number of compositions (partitions where account is taken of the order of the parts) of all numbers between $\underline{X}$ and $\overline{X}$ into $n$ parts greater than or equal to zero and less than or equal to $m$ and such that at least $\nu$ parts will be at least equal to $\mu$.

An interesting quantity is the minimum total vote with which a majority of the districts can be won. We have $\underline{X}=\mu\nu=(m+1)(n+1)/4$ and $\underline{X}/mn=\frac{1}{4}+(m+n+1)/4mn$. Evidently the minimum proportion of the total vote is larger than $\frac{1}{4}$ but approaches this value as $m$ and $n$ become large. For values of $m$ and $n$ of the order usually encountered in elections, it is thus possible, although obviously unlikely, that a party may win a majority of the districts with only slightly more than one quarter of the popular vote. The result may be generalized. In order to win more than $rn$ of the districts, a party must get at least $([m/2]+1)$ votes in at least $([rn]+1)$ districts, where the square bracket indicates the largest integer less than or equal to its argument. The minimum proportion of the total vote will be $([m/2]+1)([rn]+1)/mn$, a quantity which is greater than $r/2$ and approaches it with increasing $m$ and $n$. Hence it is possible to win an $r$-majority (for example a 2/3 majority) with only slightly more than $r/2$ (for example 1/3) of the popular vote.

**3. Three districts.** For small values of $m$ and $n$, the desired probability can be found by classical enumerative methods. Since there are $m+1=2\mu$ equally likely outcomes in each of $n$ independent districts, there are $(2\mu)^n$ equally likely outcomes for the whole election. If we can enumerate $F(m, n)$, the number of favorable cases for one of the parties, the required probability will be given by $P(m, n)=2F(m, n)/(2\mu)^n$. Various systematic arrangements of the favorable cases may be used to evaluate $F(m, n)$. A convenient method is to arrange the favorable outcomes according to the party's total vote, which varies from $\underline{X}$ to $\overline{X}$. If we designate by $F_k(m, n)$ the number of favorable cases for a total vote of $\underline{X}+k$ ($k=0, 1, 2, \cdots, D$ where $D=\overline{X}-\underline{X}=\mu\nu-\mu-\nu$), then $F(m, n)=\sum_{k=0}^{k=D} F_k(m, n)$.

These observations enable us to find a simple formula for $P(m, 3)$. For $n=3$, we have $\nu=2$, $\underline{X}=2\mu$, $\overline{X}=3\mu-2$, and $D=\mu-2$. For a favorable case the party must win just two of the districts. Consider a favorable case in which the party's total vote is $2\mu+k$. We can achieve a favorable distribution of these votes by assigning $\mu$ votes to each of two of the districts and then distributing the remaining $k$ votes in any way over the three districts. The two districts can be chosen in 3 ways. The number of ways of distributing $k$ votes to 3 districts is $\binom{2+k}{k}$.* Since $k$ is always less than $\mu$ for the favorable cases, none of these distributions can lead either to duplication in counting or to the assignment of more than $m$ votes to a district. Hence

$$F_k(m, 3) = 3\binom{2+k}{k} \quad \text{and} \quad F(m, 3) = 3\sum_{k=0}^{k=\mu-2}\binom{2+k}{k}.$$

* W. A. Whitworth, Choice and Chance, N. Y., 1927, p. 97.

By a well known identity,* the summation reduces to $\binom{\mu+1}{3}$. It follows that

$$P(m, 3) = \frac{2 \cdot 3\binom{\mu+1}{3}}{(2\mu)^3}$$

$$= \frac{\mu^2 - 1}{8\mu^2}. \tag{1}$$

**4. The general case.** Pleasing arrays result from the arrangement of the favorable cases according to the total vote and the number of districts having $\mu$ or more votes. But for all except the smallest values of $m$ and $n$ it is more convenient to use a method originally due to Euler. Consider the generating function $f(z) = (1+z+z^2+z^3+ \cdots +z^{m-1}+z^m)^n$. In the expansion of $f(z)$, before collecting terms, each term arises from a choice of one term from each of the $n$ factors. If we let the choice of $z^{x_j}$ in the $j$th factor correspond to a vote of $x_j$ in the $j$th district, there will be a one-to-one correspondence between the election outcomes and the terms of the uncollected expansion. We wish to enumerate those terms in which the exponents of $z$ are less than $\overline{X}$ and which at the same time are obtained by choosing from the $n$ factors at least $\nu$ terms whose exponents are at least $\mu$. The generating function may be rewritten as follows:

$$f(z) = \sum_{p=0}^{p=n} \binom{n}{p} \left(\frac{1 - z^\mu}{1 - z}\right)^n z^{p\mu}. \tag{2}$$

This form of $f(z)$ corresponds to an arrangement of the election outcomes according to the number of districts in which the party has a majority. The terms arising from a given $p$ correspond to outcomes in which exactly $p$ of the districts have $\mu$ or more votes. We wish to count those terms which arise from $p \geqq \nu$ and whose exponents are less than or equal to $\overline{X}$. In fact,

$$F(m, n) = \sum_{s=\underline{X}}^{s=\overline{X}} C_s,$$

where $C_s$ is defined by

$$\sum_{p=\nu}^{p=n} \binom{n}{p} \left(\frac{1 - z^\mu}{1 - z}\right)^n z^{p\mu} = \sum_{s=\mu\nu}^{s=mn} C_s z^s. \tag{3}$$

In order to evaluate $C_s$, we consider

$$\left(\frac{1 - z^\mu}{1 - z}\right)^n = \sum_{r=0}^{r=(\mu-1)n} H_r z^r. \tag{4}$$

A comparison of (4) and (3) shows that $C_s = \sum \binom{n}{p} H_{s-p\mu}$, where the summation

* $\binom{A+B}{B} = \sum_{s=0}^{s=A} \binom{B+s+1}{s}$. (*Ibid.*, p. 79.)

extends over $p$ satisfying $\nu \leqq p \leqq s/\mu$. Hence

$$F(m, n) = \sum_{s=\mu\nu}^{s=\overline{X}} \sum_{p=\nu}^{p=[s/\mu]} \binom{n}{p} H_{s-p\mu}, \tag{5}$$

where the square bracket indicates the greatest integer contained in the argument. Reversing the order of summation in (5) and introducing new subscripts defined by $p=\nu+i$ and $s-p\mu=r$, we find

$$F(m, n) = \sum_{i=0}^{i=[D/\mu]} \binom{n}{\nu+i} \sum_{r=0}^{r=D-i\mu} H_r. \tag{6}$$

The summation involving $H_r$ may be evaluated by expanding (4) in the form

$$(1-z^\mu)^n(1-z)^{-n} = \left\{\sum_{j=0}^{j=n} (-1)^j \binom{n}{j} z^{j\mu}\right\} \left\{\sum_{t=0}^{t=\infty} \binom{n+t-1}{t} z^t\right\}. \tag{7}$$

The desired terms may be obtained by multiplying the first term in the first bracket by the first $D-i\mu$ terms in the second bracket, the second term in the first bracket by the first $D-(i+1)\mu$ terms of the second bracket, and so on, the process stopping with the last term in the first bracket for which $j\mu \leqq D-i\mu$. Hence

$$\sum_{r=0}^{r=D-i\mu} H_r = \sum_{j=0}^{j=[D/\mu]-i} (-1)^j \binom{n}{j} \sum_{t=0}^{t=D-(i+j)\mu} \binom{n+t-1}{t}. \tag{8}$$

The summation over $t$ reduces to $\binom{n+D-(i+j)\mu}{n}$ by means of the identity used in Section 3. With this simplification, (6), (8), and the remarks at the beginning of Section 3 imply that $P(m, n)$ is given by the expression

$$\frac{1}{2^{n-1}\mu^n} \sum_{i=0}^{i=[D/\mu]} \sum_{j=0}^{j=[D/\mu]-i} (-1)^j \binom{n}{j} \binom{n}{\nu+i} \binom{n+D-(i+j)\mu}{n}. \tag{9}$$

It can easily be verified that (9) reduces to (1) for $n=3$. For $m=3$, it reduces to

$$P(3, n) = \frac{1}{2^{2n-1}} \sum_{i=0}^{i=[\nu/2]-1} \sum_{j=0}^{j=\nu-2(i+1)} \binom{n}{\nu+i} \binom{n}{j}. \tag{10}$$

For particular values of $m$ and $n$, other special formulas may be derived, but none of these results are convenient for large values of the arguments.

**5. Large number of voters in each district.** Let $u_j=x_j/m$, where $x_j$ is defined as in Section 2. Then $u_j>\frac{1}{2}$ is equivalent to a majority in the $j$th district, and $\sum_1^n u_j<n/2$ is equivalent to a minority total vote. Let $P_k$ be the probability of a favorable case in which $u_j>\frac{1}{2}$ for $j=1, 2, \cdots, k$, and $u_j<\frac{1}{2}$ for $j>k$. Since we can choose $k$ districts in $\binom{n}{k}$ ways, $\binom{n}{k} P_k$ is the probability of the favor-

able cases in which just $k$ of the districts have a majority. Hence

$$P(m, n) = 2\sum_{k=r}^{k=n-1}\binom{n}{k}P_k, \tag{11}$$

where $P_k$ is the probability that the variables determine a point lying in the region $R_k$ defined by $\sum_1^n u_j < n/2$, $\frac{1}{2} < u_j \leqq 1$ $(j \leqq k)$, $0 \leqq u_j < \frac{1}{2}$ $(j > k)$. As $m$ becomes large, the probability law of the $u_j$ approaches the continuous uniform distribution between zero and one. Hence $P_k$ approaches the volume of the $n$ dimensional region $R_k$, and $P(\infty, n)$ will be given by (11) with $P_k$ replaced by the volume of $R_k$.

In order to find the volume of $R_k$ we perform the translation $u_j = \alpha_j + \frac{1}{2}$ $(j \leqq k)$, $u_j = \alpha_j$ $(j > k)$. The equations defining the region become $\sum_1^n \alpha_j < (n-k)/2$, $0 \leqq \alpha_j \leqq \frac{1}{2}$ for all $j$. Evidently $R_k$ is now the region under the hyperplane with intercepts $(n-k)/2$ and within the hypercube bounded by the axis hyperplanes and the hyperplanes $\alpha_j = \frac{1}{2}$. The planes $\alpha_j = \frac{1}{2}$ divide the region into a number of similar regions. Let $V_i$ $(i = 0, 1, 2 \cdots, n-k-1)$ stand for the volume of the region given by $\sum_1^n \alpha_j < (n-k)/2$, $\alpha_j > \frac{1}{2}$ for $j \leqq i$, and $\alpha_j > 0$ for $j > i$. There are $\binom{n}{i}$ equal regions of this type in each of which *at least* $i$ of the $\alpha_j$ exceed $\frac{1}{2}$. We desire the volume of the region under the plane for which none of the $\alpha_j$ exceeds $\frac{1}{2}$. The situation is exactly analogous to that covered by Whitworth's Proposition XIV.* Hence we may write

$$\text{Vol. } R_k = V_0 - nV_1 + \binom{n}{2}V_2 - \cdots + (-1)^i\binom{n}{i}V_i + \cdots \pm \binom{n}{n-k-1}V_{n-k-1}. \tag{12}$$

To evaluate $V_i$, we move the origin to the point $\alpha_j = \frac{1}{2}$ for $j \leqq i$ and $\alpha_j = 0$ for $j > i$. The region is now bounded by the axis hyperplanes and the hyperplane with intercepts $(n-k-i)/2$. A simple iterated integration shows that the volumes is given by $(n-k-i)^n/2^n n!$ Substituting this value for $V_i$ in (12), we have

$$\text{Vol. } R_k = \frac{1}{2^n n!}\sum_{i=0}^{i=n-k-1}(-1)^i\binom{n}{i}(n-k-i)^n.\dagger \tag{13}$$

Replacing $P_k$ in the right member of (11) by this expression for Vol. $R_k$ we get a formula for $P(\infty, n)$ which, after the introduction of new indices of summation

* *Ibid.*, p. 73 ff.

† This result may be found analytically by constructing, with the aid of Fourier integrals, a function which takes the value 1 in the region $R_k$ and zero elsewhere. Integration of this function over the whole space yields (13).

defined by $r=n-k-i$ and $s=i$, reduces to

$$P(\infty, n) = \frac{1}{2^{n-1}n!} \sum_{r=1}^{r=r-1} \sum_{s=0}^{s=\mu-1-r} (-1)^s \binom{n}{s} \binom{n}{s+r} r^n. \tag{14}$$

**6. Large numbers of districts.** In order to find $P(m, \infty)$ we apply a central limit theorem to the $X$ and $Y$ of Section 2. We consider the normalized variables

$$\xi_n = \frac{X - \mathcal{E}(X)}{\sigma_X} \quad \text{and} \quad \eta_n = \frac{Y - \mathcal{E}(Y)}{\sigma_Y},$$

where $\mathcal{E}(\ )$ signifies the mathematical expectation and $\sigma$ stands for the standard deviation. It can easily be shown that $R$, the correlation coefficient of $\xi_n$ and $\eta_n$, is independent of $n$. In fact it equals the correlation coefficient of $x_j$ and $y_j$. The definition of $R$ and the probability law of the $x_j$ assumed in Section 2 yield

$$R = \frac{\mu\sqrt{3}}{\sqrt{4\mu^2 - 1}}. \tag{15}$$

From a central limit theorem due to Bernstein* we know that as $n \to \infty$ the joint elementary probability law of $\xi_n$ and $\eta_n$ tends uniformly to the law

$$\Phi(\xi, \eta) = \frac{1}{2\pi\sqrt{1 - R^2}} e^{-(\xi^2 - 2R\xi\eta + \eta^2)/2(1-R^2)}. \tag{16}$$

Hence the probability of the point $(\xi_n, \eta_n)$ lying in any region will approach the integral of $\Phi$ over the region. But from the probability law assumed for $x_j$ and the definition of $y_j$ it follows that $\mathcal{E}(X) = n\mathcal{E}(x_j) = mn/2$ and $\mathcal{E}(Y) = n\mathcal{E}(y_j) = n/2$. Hence $\xi_n < 0$ is equivalent to a minority total vote, and $\eta_n > 0$ is equivalent to a majority of the districts. Hence $P(m, n)$ is the probability of $\xi_n/\eta_n < 0$, and $P(m, \infty)$ is the integral of $\Phi$ over the region $\xi/\eta < 0$. The integration yields

$$P(m, \infty) = \frac{1}{2} - \frac{1}{\pi} \arctan (R/\sqrt{1 - R^2}). \tag{17}$$

Making use of (15), we have

$$P(m, \infty) = \frac{1}{2} - \frac{1}{\pi} \arctan (\mu\sqrt{3}/\sqrt{\mu^2 - 1}). \tag{18}$$

**7. Summary.** In the following table are displayed numerical values of $P(m, n)$ for certain values of $\mu$ and $\nu$. The numbers in parentheses indicate the formulas used for the computations.

* S. N. Bernstein, Theory of Probability, Moscow, 1934 (in Russian), pp. 241 ff., 317 ff. Bernstein gives a simple sufficient condition on the moments which is easily seen to be satisfied in this case. See, alternatively, J. V. Uspensky, Introduction to Mathematical Probability, N. Y., 1937, pp. 318 ff.

The stability of $P(m, n)$ is striking. It appears to be an increasing function of its arguments, but the slow rate of increase suggests that it would be difficult to prove this conjecture. For moderate values of $m$ and $n$ it is sensibly equal to 1/6. It is hard to say how more realistic hypotheses would affect the results.

| $\mu$ \ $\nu$ | 2 | 3 | 4 | 5 | ⋯ | $\nu$ | ⋯ | ∞ |
|---|---|---|---|---|---|---|---|---|
| 2 | 3/32 | .117 | .126 | .131 | ⋯ | (10) | ⋯ | .148 |
| 3 | .111 | .132 | .140 | .145 | | | | .159 |
| 4 | .117 | .137 | .145 | .149 | | | | .162 |
| 5 | .120 | .139 | .147 | .151 | | | | .164 |
| ⋮ | ⋮ | | | | | ⋮ | | ⋮ |
| $\mu$ | (1) | | | | ⋯ | (9) | ⋯ | (18) |
| ⋮ | ⋮ | | | | | ⋮ | | ⋮ |
| ∞ | 1/8 | .143 | .150 | .154 | ⋯ | (14) | ⋯ | 1/6 |

Dropping the assumption of independence should make the probability smaller, but the introduction of a more centrally concentrated probability distribution would make it larger, since unfair outcomes are more likely for nearly even elections. On the basis of the simple case here considered, the best we can do is to suggest that for the values of $m$ and $n$ usually encountered the probability is of the order of 1/6.

# THE QUADRILATERALS OF PASCAL'S HEXAGRAM

W. H. BUNCH, Nyssa, Oregon

**1. Introduction.** Pascal's hexagram has inspired more interest and investigation than almost any other geometric configuration. Many intriguing figures have been found and brought from their hiding places in the background of this hexagram. Pascal's famous theorem states that if a hexagon is inscribed in a conic the three pairs of opposite sides meet in three points on a straight line. This theorem was first published by Pascal on a little placard in 1640 when he was sixteen years old [1]. This line is known as Pascal's line, and we shall call the points on the line Pascal points.

Steiner was the first to direct attention to the complete hexagram formed by joining the six points on a conic in every way. In this there are sixty Pascal lines passing by four's through each of the forty-five Pascal points. Steiner also showed that the sixty Pascal lines pass by three's through twenty $g$ points which may be divided into ten pairs such that the members of each pair are harmonic conjugates with respect to the conic, and these $g$ points lie four by four on fifteen lines [2].

Kirkman extended Steiner's theorem to show that the Pascal lines intersect three by three, not only in Steiner's twenty $g$ points but also in sixty other $h$ points [3]. Cayley and Salmon showed that there are twenty $G$ lines, each of which pass through one $g$ and three $h$ points, and that these $G$ lines pass four by four through fifteen $i$ points. Various proofs and demonstrations of these theorems on Pascal's hexagram as well as additional properties have been discussed by Salmon [4], Cayley [5], Veronese [6], Cremona [7], and Christine Ladd [8].

Brianchon developed the dual of Pascal's theorem by the method of polar reciprocals. His theorem states that the three lines joining the three pairs of opposite vertices of a hexagon circumscribed about a conic meet in a point. This point is the pole of a Pascal line and is known as a Brianchon point. The three lines are the polars of three Pascal points on a line and we shall call them Brianchon lines. The complete figure connected with this theorem contains sixty Brianchon points and forty-five Brianchon lines. Ann and Elizabeth Linton made some remarkable drawings of the results mentioned above [9], and the author made a study of the hexagram with certain elements in motion, and used the machinery thus set up to make certain straight line constructions on unicursal cubics [10].

The purpose of this paper is to show that the 60 Pascal lines form 15 distinct quadrilaterals whose vertices are Pascal points and whose diagonals are the 45 Brianchon lines. We shall also show the relationship of these quadrilaterals to the results obtained by others.

**2. Notation.** We shall designate the six vertices of the hexagon by the digits 1, 2, 3, 4, 5, and 6. The notation, 12, will mean the line joining points 1 and 2,

and 12,45 will be the intersection of lines 12 and 45. Any such combination representing the intersection of nonconsecutive sides will be a Pascal point. If we choose any permutation of the six digits, say, 354126, then pair the first and second with the fourth and fifth positions, 35,12, and the second and third with the fifth and sixth, 54,26, and the third and fourth with the sixth and first positions, 41,63, we have three points on a Pascal line. We shall, therefore, designate a Pascal line by the permutation which locates the three points on that line.

**3. The main theorems.**

THEOREM 1. *The sixty Pascal lines are divided into fifteen sets of four lines, such that each set forms a quadrilateral whose six vertices are Pascal points, and the quadrilaterals are distinct in the sense that no Pascal line is used to form the side of more than one quadrilateral.*

Proof. Consider the Pascal lines, (a) 123456, (b) 213546, (c) 132465, and (d) 624351. We find that the six points 12,45; 23,56; 34,61; 13,46; 35,62; and 24, 51 are the only points on all four lines. Also two and only two of these lines meet in each point. Hence the four lines form a quadrilateral whose vertices are the given points. We shall call this quad. $A$.

This accounts for two Pascal lines through each point, but there are four. We need now to investigate the other two lines through any vertex, and we choose 12,45. These lines are (e) 123546 and (f) 213456. The two lines which form a quadrilateral with these are (g) 132564 and (h) 625341. We shall call this quad. $B$. If we investigate quad. $B$ in the same manner as we did quad. $A$, we find that these two quadrilaterals have only one vertex in common. The similarity of the hexagram now shows that the six vertices of a quadrilateral serve, one each, as the vertices of six other separate quadrilaterals. Each Pascal line is therefore used in only one quadrilateral.

We shall call this group of fifteen quadrilaterals the $\alpha$ collection. The six quadrilaterals which have one vertex each in common with quad. $A$, we shall call the $\alpha_1$ collection for quad. $A$. We shall call the eight which have no common vertices with quad. $A$ the $\alpha_2$ collection for quad. $A$. We shall also define a closed set of quadrilaterals to mean a set of three quadrilaterals, any two of which have only one common vertex, and all three of which have a pair of opposite vertices confined to three points.

THEOREM 2. *Any quadrilateral of the $\alpha$ collection forms three closed sets with the quadrilaterals of its $\alpha_1$ collection, and the $\alpha$ collection contains fifteen closed sets.*

Proof. The four lines (i) 314625, (j) 341652, (k) 651342, and (l) 615324 form a quadrilateral which we shall call quad. $C$. Quads. $A$ and $B$ have the common point 12,45. Quads. $B$ and $C$ have the common point 14,25, and $A$ and $C$ have 15,24 in common. The first two of these points do not lie on the same Pascal line, hence they are opposite vertices of quad. $B$. Likewise the second and third

are opposite for quad. $C$ and the first and third for quad. $A$. Therefore, quads. $A$, $B$, and $C$ form a closed set. We may notice that the three points that determine a closed set are formed by the same four digits, and that any two are formed from the third by permuting pairs of digits across the comma. The number of combinations of six things used four at a time is fifteen, which is the number of closed sets possible in the $\alpha$ collection. Since each quadrilateral has three pairs of opposite vertices it is a member of three closed sets.

THEOREM 3. *The 45 diagonals of the 15 quadrilaterals of the $\alpha$ collection are the 45 Brianchon lines which are the duals of the 45 Pascal points.*

Proof. The proof of this theorem follows almost immediately from the theorem from projective geometry on the inscribed quadrilateral. If we choose the quadrilateral whose vertices are the points 1, 2, 4, and 5, then the pairs of tangents through 2 and 5, and 1 and 4 meet on a line through 15,24 and 12,45. But this line is both a Brianchon line and a diagonal of quad. $A$.

Now we shall define an open set of quadrilaterals to be a set of three, no two of which have common vertices, but the nine diagonals of the set must meet by three's in three points such that one diagonal from each quadrilateral passes through each point.

THEOREM 4. *Each quadrilateral of the $\alpha$ collection is a member of four open sets and there are twenty such sets in the $\alpha$ collection.*

Proof. We shall introduce a notation here that we shall call a dual notation. In this notation the six digits represent the six tangents to the conic at the respective vertices of the hexagon. Then the notation, 123456, gives the three Brianchon lines through a point; these lines are the duals of the three Pascal points on a line determined by the same notation. Choose now a pair of opposite vertices on quad. $A$, such as 12,45 and 24,51. If we apply the theorem on the inscribed quadrilateral again, we find the diagonal joining these points to be, in dual notation, 25,41. This is exactly the same as the standard notation for the third Pascal point which, with the two above, determines a closed set of quadrilaterals. By reversing this scheme we may find a pair of opposite vertices on a quadrilateral from a given diagonal. Then by writing the Pascal lines through these, and selecting the ones that have common Pascal points, we can find a quadrilateral from a given diagonal.

Starting in dual notation with the diagonal of quad. $A$, that is, 52,41, we build the permutation 143256 which gives three diagonals that meet in a point. Each diagonal is from a different quadrilateral. The first, 14,25, is from quad. $A$, the second, 43,56, is from one we shall call quad. $D$, and 32,61 is from quad. $E$. In the following table the first three rows give these three quadrilaterals in standard notation. Rows 4, 5, and 6 give their vertices with their opposite vertices paired in parentheses. Rows 7, 8, and 9 give their diagonals in dual notation with each set of diagonals directly below the vertices they connect. Row 10 gives the permutations which show that these diagonals meet in three points.

TABLE I

| | | | | |
|---|---|---|---|---|
| 1. | Quad. *A*. | 123456, | 213546, | 132465, 624351. |
| 2. | Quad. *D*. | 451362, | 541632, | 461352, 641532. |
| 3. | Quad. *E*. | 314265, | 134625, | 364215, 634125. |
| 4. | Quad. *A*. | (12, 45−24, 51), | (23, 56−35, 62), | (34, 61−13, 46). |
| 5. | Quad. *D*. | (45, 36−64, 53), | (24, 13−41, 32), | (16, 25−15, 26). |
| 6. | Quad. *E*. | (31, 26−63, 21), | (15, 46−14, 56), | (24, 53−25, 34). |
| 7. | Quad. *A*. | 14, 25, | 36, 25, | 14, 36. |
| 8. | Quad. *D*. | 43, 56, | 43, 12, | 56, 12. |
| 9. | Quad. *E*. | 32, 61, | 54, 16, | 45, 32. |
| 10. | | 143256, | 634521, | 145632. |

A comparison of the vertices in rows 4, 5, and 6 of the table shows that quads. *A*, *D*, and *E* have no common vertices. And since their diagonals meet in three points such that one diagonal from each quadrilateral passes through each of the three points, they form an open set. Since the points where the diagonals meet are the Brianchon points, four of these points lie on each diagonal and each quadrilateral is a member of four open sets. Quads. *D* and *E* are members of the $\alpha_2$ collection for quad. *A*; hence we may state that the $\alpha_2$ collection is divided into four pairs and that each pair forms an open set with quad. *A*. Since there are sixty Brianchon points and three are used in each open set, there are twenty open sets in the $\alpha$ collection of quadrilaterals.

We shall now show a correspondence between these sets of quadrilaterals and the results obtained by Steiner, Kirkman, Cayley, and Salmon. Salmon used the notation,

$$\begin{Bmatrix} 12.45.36 \\ 65.23.14, \\ 34.16.25 \end{Bmatrix}$$

to represent the three Pascal lines, 123654, 163452, and 143256, which intersect in a Steiner *g* point [4]. We notice that each row and each double column contains all six digits in such a way that each two rows represent a Pascal line. The dual notation for the three Brianchon points in Table I, row 10, is the same as these. Hence the nine diagonals of the quadrilaterals in an open set intersect by three's in three points on a straight line, and this line is the polar of a Steiner *g* point.

The notation for a Kirkman *h* point is

$$\begin{Bmatrix} 41.35.62 \\ 52.64.31. \\ 63.51.24 \end{Bmatrix}$$

Here each row and the first column only contain all six digits without repetition, and the three Pascal lines are 146253, 251364, and 415362. The three *h* points on a Salmon-Cayley *G* line are the ones just given and

$$\begin{Bmatrix} 65.13.42 \\ 34.26.15 \\ 21.35.46 \end{Bmatrix} \text{ and } \begin{Bmatrix} 16.42.53 \\ 23.51.46. \\ 54.26.31 \end{Bmatrix}$$

The notation for the Steiner point on this line is set up by using the first column from each of these. We must also notice that the lines 14,52 and so forth, which make up the first columns, are not duplicated in the notation for any of the $h$ points. Six of the fifteen lines joining the six points on a conic are used in forming any Pascal line. Three Pascal lines are formed without the use of these six. The three thus formed meet in an $h$ point which is said to correspond to the Pascal line formed by the first six.

In the following table the capital letters represent the quadrilaterals of the $\alpha_2$ collection for quad. $A$. The first column of six digits is composed of the Pascal lines which make up those quadrilaterals. The second column contains the $h$ points which correspond to the Pascal lines. The small letters with subscripts are used to identify Pascal lines and rows in the notation for a Kirkman $h$ point.

TABLE II

| | | | | | |
|---|---|---|---|---|---|
| $A$ | $i_2$ | | $41.35.62-a_1$ | $g_1$ | |
| | $h_3$ | 123456 | $52.64.31-b_1$ | $g_2$ | 132465 |
| | $g_4$ | | $63.51.24-c_1$ | $i_4$ | |
| | $i_1$ | | $52.34.61-a_2$ | $h_1$ | |
| | $g_3$ | 213546 | $41.65.23-b_3$ | $h_2$ | 624351 |
| | $h_4$ | | $36.42.15-c_2$ | $i_3$ | |
| $D$ | $f_1-$ | 451362 | $43.61.52$ | $a_1-$ | 314265 |
| | | | $65.32.41$ | | |
| | | | $12.46.35$ | | |
| | | 541632 | $65.13.24-g_1$ | $b_1-$ | 134625 |
| | | | $34.26.15-h_3$ | | |
| | | | $21.35.46-i_1$ | | |
| | $d_1-$ | 461352 | $43.51.62$ | $c_1-$ | 364215 |
| | | | $56.32.41$ | | |
| | | | $12.45.36$ | | |
| | $e_1-$ | 641532 | $65.31.42$ | | 634125 |
| | | | $34.52.61$ | | |
| | | | $12.63.54$ | | |

| | | | | | |
|---|---|---|---|---|---|
| $A$ | $g_1$ | | $41.62.35-a_3$ | | |
| | $g_2$ | 132465 | $63.45.12-b_2$ | | |
| | $i_4$ | | $52.16.43-c_3$ | | |
| | $h_1$ | | $36.45.12-a_4$ | | |
| | $h_2$ | 624351 | $52.13.46-b_4$ | | |
| | $i_3$ | | $14.56.23-c_4$ | | |
| $E$ | $a_1-$ | 314265 | $32.64.15$ | | |
| | | | $61.25.34$ | | |
| | | | $45.36.21$ | | |
| | $b_1-$ | 134625 | $16.24.35$ | | |
| | | | $23.65.14$ | | |
| | | | $45.12.63$ | | |
| | $c_1-$ | 364215 | $32.14.65$ | | |
| | | | $16.25.34$ | | |
| | | | $45.31.26$ | | |
| | | 634125 | $23.51.46-e_1$ | | |
| | | | $16.42.35-d_1$ | | |
| | | | $45.13.62-f_1$ | | |

A study of Table II shows a certain correspondence. The digits in the rows $a_1$, $b_1$, and $c_1$ of the $h$ point correspond to Pascal 123456 in quad. $A$, and they have the same permutations, respectively, as the Pascal lines $a_1$, $b_1$, and $c_1$ in quad. $E$. The fourth line in quad. $E$ has no such correspondence, but its corresponding $h$ point has the same relationship to Pascal lines $d_1$, $e_1$, and $f_1$ of quad. $D$. Again, the $h$ point corresponding to the fourth line of quad. $D$ has rows, $g_1$, $h_1$, $i_1$, whose permutations are the same as those of lines $g_1$, $h_1$, and $i_1$ of quad. $A$. This open set of quadrilaterals forms a complete cycle of these correspondences. The three $h$ points thus chosen lie on the same $G$ line, and the $g$ point formed by their first columns lies on the same line. Also the nine diagonals

of this open set are concurrent by three's on the polar reciprocal of this same $g$ point. We also find that one $h$ point from each quadrilateral of this open set lies on the same Pascal line through this $g$ point, and there are three such groupings in the set. Hence the twelve $h$ points corresponding to the twelve Pascal lines in an open set lie by three's on four lines through a $g$ point. We have thus set up a direct correspondence between an open set of quadrilaterals and a $G$ line and also a $g$ point.

By making a table similar to Table II for the entire $\alpha_2$ collection, we find that quad. $A$ holds the same relationship to each of the four open sets in its $\alpha_2$ collection. The four $G$ lines corresponding to these four sets meet in an $i$ point. The four $g$ points, however, do not lie on an $I$ line. (Salmon's proof is sufficient [4].) Now if we examine the twelve Pascal lines that make up the open set $A$, $D$, $E$, we find that the three that are not identified in the Table by a letter with subscript, 1, meet in a $g$ point. These are the three that correspond to three $h$ points on a $G$ line. We also find the other nine lines divided into three collections of three each, such that lines $i_1$, $f_1$, and $c_1$ meet in an $h$ point as do the collections $g_1$, $e_1$, $a_1$, and $h_1$, $b_1$, $d_1$. These four points determine a $G$ line. The corresponding collections of lines determined by the other open sets, identified by subscripts 2, 3, and 4, respectively, each determine a $G$ line in the same way. The four $g$ points so determined lie on an $I$ line but the four $G$ lines do not pass through an $i$ point.

We may now state that the four open sets of an $\alpha_2$ collection for any quadrilateral determine four $g$ points on an $I$ line and correspond to four $G$ lines through an $i$ point. There are fifteen quadrilaterals, fifteen $\alpha_2$ collections, fifteen $I$ lines, and fifteen $i$ points. Again, since any open set of three quadrilaterals is a member of three $\alpha_2$ collections and each collection determines a different $I$ line and corresponds to a different $i$ point, while the set in each case determines the same $g$ point and corresponds to the same $G$ line, there are three $I$ lines through each $g$ point and three $i$ points on each $G$ line. It might be of interest to add that the $g$ point determined by an open set and the one corresponding to an open set are harmonic conjugates with respect to the conic.

In order to make a further study of the $\alpha$ collection we shall designate a quadrilateral by a letter and by the dual notation for one of its diagonals. For the members of the $\alpha_2$ collection we shall choose the diagonal that passes through one of the four Brianchon points on diagonal 14,25 of quad. $A$. The $\alpha_2$ collection, paired as suggested above then becomes: ($D-43,56$; $E-32,61$), ($F-13,56$; $G-32,64$), ($D_1-35,64$; $E_1-13,26$), and ($F_1-35,61$; $G_1-34,26$). We now notice a different grouping of four pairs. The notations for quads. $D$ and $D_1$ are formed by the same four digits; hence they have a common vertex and the diagonals here represented both come from that vertex. The same is true of the other pairs, $E$, $E_1$ and $F$, $F_1$ and $G$, $G_1$.

If we consider the diagonals 15,42 and 12,45 of quads. $B$ and $C$, and the Brianchon lines that meet in four points on each of them, we find that the pairs

$D$, $E_1$ and $D_1$, $E$ meet on 15, 42. Hence, the four lines $D$, $E_1$, $D_1$, and $E$ form a quadrilateral, two nonconsecutive vertices of which are Pascal points; two are Brianchon points on 14,25; and two are Brianchon points on 15,42. Similarly we may show that $F$, $G$, $F_1$, and $G_1$ form a quadrilateral with two vertices in Pascal points, two in Brianchon points on 14,52 and two on 12,54. The other four Brianchon points, two on 15,42 and two on 12,54, also serve as vertices for a similar quadrilateral and the other two vertices are Pascal points. The six Pascal points used in these three quadrilaterals all lie on line 36. Line 36 is the dual of point 36 and the three quadrilaterals above with their properties may be obtained by dualizing the closed set of quads. $A$, $B$, and $C$, and by employing Theorems 5 and 6.

THEOREM 5. *The three diagonals, one from each quadrilateral of a closed set, which passes through the three points which are each common vertices of two members of the set, form a triangle in which each vertex is the pole of the opposite side with respect to the given conic.*

Proof. The vertices of the triangle formed by the diagonals of quads. $A$, $B$, and $C$ are the Pascal points 12,45, 24,51, and 25,41. The dual notation for any one of these points gives the Brianchon line which passes through the other two. This follows from the theorem on the inscribed quadrilateral.

THEOREM 6. *The six diagonals, two from each quadrilateral of a closed set, which do not pass through a vertex common to two members of the set, are concurrent.*

Proof. In dual notation, the diagonals of quad. $A$ which are determined by vertices which are not common to a vertex of quads. $B$ or $C$ are 25,36 and 14,36. Likewise those of quad. $B$, determined by vertices which are not common to a vertex of quads. $A$ or $C$, are 15,36 and 24,36, and those of quad. $C$ are 12,36 and 45,36. We now notice that these diagonals all pass through the point 36.

The diagonals of quad. $A$ are 14,25, 14,36, and 36,25. If one of these is given we may separate the notation for it at the comma, and use the two digits not used in its notation with each half to form the notation for the other two. Thus if the given diagonal is 14,25, the other two must contain the point 36 in their notations. Since the notation for the diagonals of Theorem 5 are formed by the same four digits, the notation for the other six diagonals in the set must each contain the other two digits in such a way as to represent a point.

This theorem sets up a one to one correspondence between the fifteen closed sets and the fifteen points 36 and 12, and so on. We may therefore designate a closed set by the notation for the point which corresponds to it.

The Kirkman $h$ point,

$$\begin{cases} 15.34.26 \\ 42.56.13, \\ 36.14.25 \end{cases}$$

corresponding to the Pascal line 123456 of quad. $A$ lies on the Pascal line 314625

of quad. $C$, and, conversely, the $h$ point,

$$\begin{cases} 36.24.15 \\ 12.65.24, \\ 45.32.16 \end{cases}$$

corresponding to the last Pascal line lies on the first. The twelve Pascal lines that make up a closed set of quadrilaterals may be divided into six pairs so related. Since quad. $A$ is a member of three closed sets, each line in quad. $A$ is paired with three other lines, one in each set. It follows that these three lines correspond to the three $h$ points on the given line of quad. $A$. Conversely, since one Pascal line corresponds to only one $h$ point, the three Pascal lines, one from each set, must pass through that $h$ point. The four lines forming quad. $A$ correspond to four $h$ points, and the twelve Pascal lines concurrent by three's in these are chosen by two's from each of the six quadrilaterals of the $\alpha_1$ collection for quad. $A$.

The results that are new in this paper are:

1. The division of the 60 Pascal lines into distinct collections of four each.
2. The relationship between these collections, or quadrilaterals.
3. The correspondence between these quadrilaterals and the results obtained by others.

**References**


1. Cuvres de Pascal par Brunschvig et Boutroux, t, I, 252. Charles, Histoire de la Géométrie, 70.
2. Crelle's Journal, vol. V, p. 274, and vol. LXII, p. 142.
3. Cambridge Dublin Mathematical Journal, vol. V, p. 185.
4. Salmon, Conic Sections, Tenth Edition, Art. 267.
5. Quarterly Journal, vol. IX, p. 348.
6. Nuovi Teoremi sull' Hexagrammum Mysticum, Memoirs, Reale Accademia dei Lincei, 1877.
7. Salmon, Geometry of Three Dimensions, Art. 536.
8. Am. Journal of Math., Vol. 2, 1879, pp. 1–12.
9. Pascal's Mystic Hexagram (University of Penn. thesis), Philadelphia, 1921.
10. This MONTHLY, vol. XL, 1933, No. 5, May, p. 251; and vol. XLII, 1935, No. 2, Feb., p. 74.

# NOTES ON COSPHERICAL POINTS

N. A. COURT, University of Oklahoma

**1. Introduction.** Some of the geometrical elements, and some of their related properties, that have been associated with the tetrahedron [1] that is, with four points in space, are here associated with $n$ cospherical points. The analogies derived are so obvious that, to save space, they will not be pointed out.

**2. Definitions.** Consider $n$ points $A_i$ $(i=1, 2, \cdots, n)$ on a sphere $(O)$, center $O$, radius $R$. Let $G$ be the centroid of the $n$ given points and $k^2$ the sum of the squares of the $n(n-1)/2$ segments determined by the $n$ points $A_i$. The line $OG$ may be called the *Euler line* of the given points $A_i$.

The centroids of the $n$ given points taken $(n-1)$ at a time form a new set of $n$ points which may be called the *medial set of points* associated with the $n$ points $A_i$. The line $A_kG_k$ joining the point $A_k$ to the centroid $G_k$ of the remaining $(n-1)$ given points may be called a *median* of the given points $A_i$. The given points $A_i$ have $n$ medians $A_kG_k$.

**3. The medial sphere.** The points $G$, $A_k$, $G_k$ are collinear and we have $GG_k:GA_k=-1:(n-1)$. Thus the set of medial points $G_k$, $k=1, 2, \cdots, n$, associated with the points $A_i$ correspond to the points $A_i$ in a homothecy $[G, -(n-1)]$ having $G$ for center and $-(n-1)$ for homothetic ratio. Since the points $A_i$ are cospherical, by assumption, we have: *The medial set of points associated with the n given cospherical points $A_i$ lie on a sphere.*

This sphere may be called the *medial sphere* of the given points. It will be denoted by $(L)$.

The point $G$ is the internal center of similitude of the spheres $(O)$, $(L)$. The center $L$ of $(L)$ lies on the Euler line $OG$ and we have

$$GO:GL = -(n-1).$$

The radius of $(L)$ is equal to $R/(n-1)$.

The harmonic conjugate $M$ of $G$ for $O$ and $L$ may be called the *Monge point* of the points $A_i$; this is the external center of similitude of the spheres $(O)$, $(L)$, and we have

$$ML:MO = 1:(n-1),$$

hence

$$(n-2)MG = 2GO.$$

**4. The $(Q)$-sphere of the given points $A_i$.** The sphere $(Q)$ having $M$ for center and coaxal with $(O)$ and $(L)$, that is, the external sphere of antisimilitude of the latter two spheres will be referred to as the *$(Q)$-sphere* of the given points $A_i$. The square of its radius will be denoted by $q$.

The two spheres $(O)$, $(L)$ are inverse with respect to $(Q)$, hence $q^2$ is equal to

the product of the powers of $M$ for these two spheres, so that

$$q^2 = (MO^2 - R^2)\frac{ML^2 - R^2}{(n-1)^2}$$

$$= (MO^2 - R^2)\frac{(n-1)^2ML^2 - R^2}{(n-1)^2};$$

hence

$$q = \frac{MO^2 - R^2}{n-1}.$$

Thus: *The square of the radius of the sphere* $(Q)$ *of the* $n$ *given points* $A_i$ *is equal to* $1/(n-1)$ *part of the power of the Monge point with respect to the circumsphere of the given points.*

**5. The $(G)$-sphere of the points $A_i$.** The sphere $(G)$ having $G$ for center and coaxial with the spheres $(O)$, $(L)$, that is, the internal sphere of antisimilitude of the latter two spheres will be referred to as the *$(G)$-sphere* of the points $A_i$. Let $g$ be the square of the radius of $(G)$.

By a calculation analogous to the one which yielded the value of $q$ we obtain

$$g = -\frac{(GO^2 - R^2)}{n-1} = -\frac{g_0}{n-1},$$

where $g_0$ denotes the power of $G$ for the sphere $(O)$. Now [2]

$$g_0 = -\frac{k^2}{n^2};$$

hence

$$g = \frac{k^2}{n^2(n-1)}.$$

**6. The sum of the powers of the given points.** Let $N_i$ be the foot of the perpendicular from the point $A_i$ upon the common radical plane $\sigma$ of the four spheres $(O)$, $(L)$, $(G)$, $(Q)$. The power $r_i$ of $A_i$ for the sphere $(Q)$ is given by the relation [3],

$$r_i = 2N_iA_i \cdot MO;$$

hence

$$\sum r_i = 2MO \cdot \sum N_iA_i.$$

Now if $N$ is the foot of the perpendicular from the centroid $G$ upon the radical

plane $\sigma$, we have

$$\sum N_iA_i = n \cdot NG;$$

hence

$$\sum r_i = 2MO \cdot n \cdot NG.$$

The two spheres $(G)$, $(Q)$ being orthogonal, the power of $G$ for the sphere $(Q)$ is equal to $g$; hence

$$g - g_0 = 2NG \cdot MO.$$

Therefore

$$\sum r_i = n(g - g_0) = -n^2g_0/(n-1) = k^2/(n-1).$$

Thus: *The sum of the powers of n cospherical points with respect to their $(Q)$-sphere is equal to $1/(n-1)$ part of the sum of the squares of the $n(n-1)/2$ segments determined by those points.*

**7. Spheres orthogonal to a fixed sphere.**

THEOREM. *The spheres having for diameters the medians of n cospherical points are orthogonal to a fixed sphere, namely, the $(Q)$-sphere of the n given points.*

The points $A_i$, $G_i$ are homologous on the two spheres $(O)$, $(L)$ with respect to the internal center of similitude $G$, hence the point $G_i$ and the diametric opposite $C$ of $A_i$ on $(O)$ are homologous points on $(L)$ and $(O)$ with respect to the external center of similitude $M$. Thus the line $MG_iC$ meets $(O)$ in the antihomologous point $D$ of $G_i$ with respect to $M$, that is, the points $D$ and $G_i$ are inverse points with respect to the sphere of antisimilitude $(Q)$ of $(O)$ and $(L)$.

The line $DA_i$ is perpendicular to $CDM$, hence $DA_i$ lies in the polar plane of $G_i$ for the sphere $(Q)$, that is, the points $A_i$, $G_i$ are conjugate with respect to $(Q)$, and therefore the sphere having $A_iG_i$ for diameter is orthogonal to the sphere $(Q)$.

**8. Systems of cospherical circles.** From the preceding proposition we derive the following corollaries.

THEOREM 1. *The spheres having for centers n given cospherical points and orthogonal to the $(Q)$-sphere of these points cut the spheres having for diameters the respective medians along n circles lying on the same sphere, namely, the $(G)$-sphere of the given points.*

Indeed, let $(A_i)$ be the sphere having $A_i$ for center and orthogonal to $(Q)$. The spheres $(A_i)$, $(G)$, $(A_iG_i)$ are orthogonal to the same sphere $(Q)$ and their centers are collinear. Hence they are coaxal, which proves the proposition.

THEOREM 2. *The polar planes of the median points of n given cospherical points with respect to the $(Q)$-sphere of the latter points cut the spheres having for diameters the respective medians along n circles lying on the same sphere, namely, the*

*circumsphere of the given points.*

Indeed, the polar plane of $G_i$ for $(Q)$ passes through the points $D$ and $A_i$ and is perpendicular to the common diametral plane $DCA_iGG_iLM$ of the two spheres $(O)$ and $(A_iG_i)$. Moreover, the points $D$ and $A_i$ belong both to $(O)$ and $(A_iG_i)$; hence the polar plane under consideration cuts the two spheres along the same circle, which proves the proposition.

THEOREM 3. *The polar planes of n given cospherical points with respect to their $(Q)$-sphere cut the spheres having for diameters the respective medians along n circles lying on the same sphere, namely, the medial sphere of the given points.*

The proof is analogous to the proof of the preceding proposition.

Note. The above propositions are applicable to $n$ concyclic points.

**References**

1. Bulletin of the American Math. Society, vol. 48, No. 8, pp. 583–588, August, 1942.
2. Nat. Math. Mag. vol. 15, 1941, p. 274 formula 8.
3. N. A. Court, Modern Pure Solid Geometry, p. 184, Macmillan, N.Y., 1935.

## A PROGRAM FOR IMPROVING THE TEACHING OF SCIENCE AND MATHEMATICS

RALEIGH SCHORLING, University of Michigan

**1. Science and Public Policy.** The suggestions in this paper stem from a recent report that was prepared for the President's Scientific Research Board. The report appears in pages 47 to 149 of the Board's fourth volume on *Science and Public Policy*, and was provided by the Cooperative Committee on Science and Mathematics Teaching. This Committee was created by representatives of several scientific societies in 1941 to work on educational problems which no single scientific group could solve as well working alone as by cooperative action. Another function of the Cooperative Committee has been to serve as a forum in which representatives of the scientific societies have been able to state the views of their own organizations, and to learn the views of other groups on the teaching of science and mathematics at the secondary and elementary levels. The Cooperative Committee is now a standing Committee of the American Association for the Advancement of Science, and is charged with the responsibility of giving attention to the problems involved in the teaching of science and mathematics at all levels.

**2. Recent report of the Cooperative Committee.*** The Cooperative Committee has made a number of reports to the parent organizations, as, for example, "A Report on High School Science and Mathematics in Relation to Manpower Problems," which was published in 1943 and distributed to more than 12,000 individuals. The most recent report, entitled "The Present Effectiveness of Our Schools in the Training of Scientists," is the one with which this paper deals. Volume Four in the series, *Science and Public Policy*, is devoted to the problem of scientific and technical manpower. In considering the sources from which future scientists may be drawn, the President's Scientific Research Board was naturally driven to consider the effectiveness of science teaching, not only in higher institutions but also in the lower schools. At this point the President's Scientific Research Board turned for aid to the Cooperative Committee.

Although the hundred-page report contributed by the Cooperative Committee is primarily concerned with the early identification of youth with special talent in science and mathematics and with the proper nurture of these gifted youth through school, college, and university, it had to give careful attention to science and mathematics in general education. In fact, the report is unique in that it is the first effort to present a picture of what is going on in a subject area throughout the educational system from the elementary school through graduate work. It indicates the major limitations of the teaching of science and mathematics at each level, and it suggests possible ways for correcting the deficiencies. The Committee spent many hours in study and discussion, and it probably comes closer to unanimous agreement on issues, whether they lie in the elementary school, in the secondary school, in the college, or in the graduate school, than is ordinarily attained by commissions.

The Committee accepted the sobering assignment of making recommendations looking toward the creation of (1) a corps of effective research scientists

---

* The present members of the Committee and the societies they represent are:
American Association of Physics Teachers: K. Lark-Horovitz, Purdue University, and Glen W. Warner, Chicago City College.
American Astronomical Society: Oliver J. Lee, Northwestern University.
American Institute of Physics: Lloyd W. Taylor, Oberlin College.
American Society of Zoologists: L. V. Domm, University of Chicago.
Botanical Society of America: Glenn W. Bladyes, Ohio State University.
Central Association of Science and Mathematics Teachers: Arthur O. Baker, Cleveland Board of Education.
Division of Chemistry Education of the American Chemical Society: Lawrence L. Quill, Michigan State College.
Executive Committee of the A.A.A.S.: E. C. Stakman, University of Minnesota.
Geological Society of America: George A. Thiel, University of Minnesota.
Mathematical Association of America: Raleigh Schorling, University of Michigan.
National Association of Biology Teachers: Prevo L. Whitaker, Indiana University.
National Council of Teachers of Mathematics: E. H. C. Hildebrandt, NorthwesternUniversity.
National Science Teachers Association: Morris Meister, Bronx High School of Science.
*Chairman*: K. Lark-Horovitz;
*Secretary*: R. W. Lefler

and (2) an organized group of discerning science educators. The report is based on the premise that our people should take such steps as may be necessary to insure (1) enough competent scientists to do whatever job may be ahead, and (2) a voting public that understands and supports the scientists' role in defense and in the design for better living. It should be noted that the second objective involves the crucial task of the education of the general public in the nature and function of science in the world of tomorrow.

This paper represents an attempt to collect the most important recommendations of the report, and to try to piece them together into a program or platform for the teaching of science and mathematics. At this point the reader is entitled to a word of caution. Lifting recommendations out of context may have done violence to the content intended by the Cooperative Committee. In any case, the critical person will wish to check the proposals in this paper with the details of the original report.

**3. Program for science and mathematics.** In developing a program for science and mathematics, this paper considers twenty major problems, described usually in the form of situations that imply limitations in our offerings in science and mathematics. First, in the treatment of each item, appears a statement of the problem; then a recommendation is given, which seems, at least to the writer, to stem from the Committee's report.

**(1).** Some of the crucial problems in the teaching of science and mathematics are much the same at all levels—elementary, secondary, and collegiate. For example, efficiency in the teaching of science and mathematics is everywhere cut down by such factors as: (a) inadequate financial support, (b) the lack of materials and space, (c) the shortage of competent teachers, (d) undesirable working conditions of teachers, (e) the unrealistic programs for teacher education, and (f) the lack of good in-service programs. Is it sensible for committees at every level and in every subject-matter area to "rediscover America" by working as if no one had every done anything on the problem?

We recommend the appointment of a National Commission on the Teaching of Science and Mathematics under the auspices of the American Association for the Advancement of Science, representing the scientists, and cooperating with agencies such as the American Council on Education, the National Education Association and the U. S. Office of Education that represent educators, administrators and science teachers (p. 59*). Presumably the suggestion here is that the Cooperative Committee be reconstituted to include representatives from other groups that are also concerned about the effectiveness of the teaching of science and mathematics in our schools. A second aim no doubt is to guarantee continuity of interest in pedagogical and curriculum problems.

* When page number only is given, the reference is to Manpower for Research, Volume Four of Science and Public Policy, a Report to the President, by John R. Steelman, Chairman, the President's Scientific Research Board, 1947. For sale by the Superintendent of Public Documents. U.S. Government Printing Office, Washington 25, D.C., at 35 cents.

**(2).** The financial support of our schools is wholly inadequate, and suggests that as a nation we are operating with a distorted sense of values. We spend about 35 billions a year on recreation, tobacco, alcohol, soap, chewing gum, beauty aids, crime, and delinquency, but only about one-tenth as much per year for education. We shall not have good teaching of science or anything else throughout the country until our people correct this situation. We are spending about 2 per cent of our annual income for education, whereas England spends a much higher percentage, and persistent rumor has it that Russia spends somewhere between 8 and 10 per cent of her annual income for education.

We should gear the financial support of public education to our economy. It should be a fixed percentage of our national income. The rate should be scientifically determined and the total amount for education should vary with our national income and with the purchasing power of the dollar. It is not sensible for us to pay without much complaint nearly 5 per cent of our national income for education in times of depression, and then recklessly allow our schools to deteriorate by spending less than 2 per cent of our income in prosperous times. Our schools need twice as many teachers as we now have and at least three times as much financial support. The writer has seen many schools, but never a good one that did not cost a lot of money.

**(3).** The nation faces a devastating shortage of teachers of science and mathematics. The main facts, widely publicized by newspapers, magazines, and radio, are: (a) There is a general shortage of teachers, (b) The situation is just about hopeless in the early grades of the elementary school. (c) Young people are rejecting teaching in the lower schools as a career. (d) Men teachers are disappearing from the high school. (e) The shortage of competent teachers of science and mathematics at the high school level appears to be more critical than in other subjects. (f) Young men who might be good material for teaching in college and graduate school are being lured into industry, business, and government by larger salaries, lighter work, and greater freedom. The report points out that in the school year 1946–47, there were about 200,000 students enrolled in 24 prominent colleges and universities that educate teachers. Of this vast student body, only 600 qualified for the teachers certificate in science and mathematics (p. 96). It is probable that in certain subjects the shortage of teachers may presently be alleviated, but the outlook as regards a sufficient number of competent teachers of science and mathematics for high schools is still dark. The study quoted above suggests that we may consider the University of Michigan as typical of large universities. In the present semester there are about 20,500 full time students at the University of Michigan. The total enrollment in the various sciences is very great. For example, we have 150 graduate students in mathematics, and about the same number in physics. We have over 4,000 elections of undergraduate mathematics courses in the college of Literature, Science, and the Arts alone. At present there are 272 seniors who are qualifying for majors in the physical and biological sciences. Moreover, the number of undergraduate and graduate students majoring in mathematics is a higher per-

centage of the total enrollment than it was 10 years ago. Now comes the shock: In student teaching, which is, of course, the gateway for teaching in high school, we have, in the present semester, the following number of student teachers: 2 in general science, 5 in mathematics, 0 in biology, 3 in chemistry, 0 in physics. Incidentally, the University of Michigan has not had a single student teacher in physics at any time during the last five semesters!

Every national scientific organization should put forth a special effort to make the public aware of the facts regarding the teacher shortage. In addition to higher salaries, we must realize that salaries and social prestige are not the only issues. For example, it is wholly unlikely that we could get or hold enough competent teachers of science merely by doubling the salaries of teachers. There are about a dozen additional things that we need to do, and most of these things relate to the working conditions of teachers. The items were listed and discussed in a document entitled "An Evolving Bill of Rights," first published in 1945. Since this statement has been reproduced frequently, we need do no more here than list the rights as follows: (a) the right to teach classes that are not too large, in general from ten to twenty pupils, (b) the right to have time in the school day for planning, (c) the right to a 45-hour week, (d) the right to adequate compensation for the full year of fifty-two weeks, (e) the right to an adequate amount of helpful and constructive supervision, (f) the right to have good materials and enough of them, (g) the right to work in a room that, with the help of students, can be made pleasant and appropriate to the tasks to be learned, (h) the right to the same personal liberties which other respectable citizens assume for themselves as a matter of course, (i) the right to an internship, (j) the right to a realistic program of in-service education, (k) the right to participate in changing the curriculum and methods, and in formulating school policies, and (l) the right to keep from being lost in the profession.*

**(4).** The undergraduate curriculum for the education of teachers, in many training institutions, is unrealistic. The report discusses this problem in a number of places. Typical limitations are: (a) Too many courses are, in fact, designed to select and train specialists in science and mathematics (p. 102). (b) Too often prospective teachers of science and mathematics for the lower schools are not trained in broad areas that will enable them to do a good job teaching several subjects in the early years of teaching in small high schools (p. 103). (c) In general, colleges and universities fail to provide a year of professionalized subject matter especially designed to meet the needs of the teachers (p. 105). (d) In science and mathematics, conventional undergraduate courses provide good background for the traditional high school courses, but do not provide adequately for the newer courses that are emerging to meet the general-education needs of pupils or for the close correlation of science and mathematics which is needed in the education of talented science students (p. 106).

---

* *Swords into Ploughshares*, p. 37. Superintendent of Public Instruction, Lansing, Michigan, 1946.

The undergraduate training program for teachers of science and mathematics must be a specifically planned professionalized program (p. 103). While the present sequence of science courses may be very satisfactory in the selection and training of future science specialists, and not too bad as a training program for teachers in large high schools, they fail to give breadth of understanding or comprehension of the interrelationships existing between the many specialized fields of science and mathematics. These understandings and relationships are essential for teachers in the secondary school, where they must, at best, teach in broad areas (p. 103). A fifth year of college training for high school teachers is clearly indicated. However, merely adding another year of the same type will not meet the needs of persons who begin their teaching in small high schools.

**(5).** At all levels, schools and higher institutions have failed to provide strong in-service programs for teachers. Supervision in the lower schools is disappearing and in too many cases has become stigmatized or has shifted from the local schools to remote institutions and to state departments of education. Yet, supervision, when provided in adequate amount and when focused on the growth of the teacher on the job, is obviously badly needed in this day when even a strong high school may each year have as much as a 25 per cent turnover of teachers, and when advances both in subject matter and professional education are great. An important phase of an in-service program is keeping materials up-to-date. School boards and administrators need to realize that curriculum revision is a technical job requiring time and energy. It can not be done by tired teachers after school.

Higher institutions, as well as public school systems, should provide strong in-service programs for teachers. Supervision should be in adequate amounts and of the type that the teacher with professional zeal wants. It should focus on teacher growth. Plans should be developed for curriculum workshops, conferences, institutes, summer school instruction, demonstrations, lectures, field trips, excursions and conferences for the training of secondary school science and mathematics teachers. We need to make provision for science and mathematics counsellors throughout the country in each of the fields of mathematics, life science, and physical science (p. 59). A bright ray in the picture as regards supervision is that an increasing number of colleges and universities, under the pressure of heavy enrollments and the consequent increases in staff, are becoming seriously concerned about training programs for young and inexperienced instructors. Incidentally, some large universities report that even the annoying problem of meager equipment can be solved partially by proper supervision of the use of such equipment as is available.

**(6).** As regards special talent in science and mathematics, our nation has not been sagacious in the use of human resources. Among the wasteful policies and practices, the report lists the following: the ratio of able youth out of college to those in college has been reported as 1 to 1. Thus we seem to lose precious

talent because many youth in normal times can not afford higher education. Then too, we have no systematic and sure method for the early identification of youngsters with a flair for science and mathematics. Nor can we be confident that the typical teacher of the early grades with little or no science training can spot a youngster with science talent, or that she would be able to design appropriate experiences that would fan the spark of genius. Then, too, high school teachers in general can not provide their students with reliable information about the science and mathematics needed on jobs and in professional careers, for two reasons: (a) the teachers lack knowledge of the applications of science and mathematics made in engineering, agriculture, mining, medicine and the like, and (b) they do not have the necessary guidance materials. The first guidance pamphlet geared to a high school subject was published in the November, 1947, issue of *The Mathematics Teacher*. Our record when identifying superior students who will be highly successful in college is poor, although several investigations suggest that it is possible now to identify them with an accuracy approximating 85 per cent (p. 124). Failure of talented youth to attend college is obviously a costly loss of valuable human resources. In too many cases, however, there is a regrettable but unnecessary waste even when talented youth do go to college, because many of the ablest of our graduate students and our young instructors in colleges and universities waste time and energy in self-support. They find themselves confronted with the dilemma of resigning themselves to a delayed entrance into the profession or struggling with an academic overload, with the almost inevitable consequence of hasty and unsatisfactory graduate work. The road to research is long, and science is a young man's game. A year or two lost can never be regained, not even if the student turns out to be a top-flight scientist.

To meet the needs of especially talented youth the school must have a continuous program of identification and guidance (p. 89). The Cooperative Committee discusses with considerable reservation such practices as rapid promotion and permission to carry extra studies. It seems to have more faith in performance on research projects at the level where the student is operating. It is also asserted that science aptitude tests, in conjunction with vocational interest inventories, will help to locate the student who should be advised of opportunities for success in fields where science is applied or research is utilized. However, the main recommendation aimed to stop the wasteful drains on our human resources is that we develop a more comprehensive record of the individual student and more nearly adequate and more competent guidance service to interpret this record intelligently. As regards the waste of precious time and energy of graduate students and young instructors, the Cooperative Committee recommends the following: "Establish Federal subsistence type scholarships for the scientifically gifted as part of a general program to support able and talented youth in all fields. This will guarantee the utmost utilization of our scientific manpower through collegiate and graduate training" (p. 59). Of special importance is the need for making financial provision that will attract and hold teachers in the

junior instructional staff. The compensation paid in the laboratories of industry and government, particularly to research workers, far exceeds the compensation of staff members even in the large universities. Basic research in the fields of all science, particularly the physical sciences, is carried on in laboratories, in industry, and in government to an extent unknown five years ago. The Committee also recommends that a large number of post-doctorate fellowships be established for: (a) junior staff type such as the National Research Council fellowships, and (b) senior staff type such as the Guggenheim fellowships (p. 59). The two purposes of the post-doctoral fellowships are (1) to utilize the ability of young scientists who have just finished the Ph.D. and are eager to continue with basic research problems, and (2) to give older scientists, who frequently have too many routine duties in their regular jobs, the opportunity to refresh their scientific spirit and to study new methods of science education and research in new environments. Science teachers need to be alerted to the drastic possibility that the proposed Science Foundation may be established without providing fellowships for talented youth. The long delay in passing a bill creating the Science Foundation has provided an opportunity to divert money, that would be needed to provide fellowships, to other projects.

**(7).** The teaching profession has not shown great resourcefulness or utilized much imagination in the design of curricula appropriate to the nurture of genius. Presumably this is a difficult task for the curriculum builder in any field. It certainly is no oversight that the report of the Cooperative Committee devotes so little space to practical suggestions as to what schools should do with youngsters who are gifted in science and mathematics. We are, therefore, in no position to be critical of the typical teacher in an elementary school or high school with little or no training in science who may not know what experiences to provide for a future scientist.

A scientific study should be undertaken to collect and perhaps to design types of experiences that should be included in a curriculum for our youth with scientific aptitudes (p. 81). It is very clear that we do not now have enough reliable information to arrive at a sensible solution of this problem. The report does suggest that a large city may provide a specialized school of science, as for example, the Bronx High School of Science in New York City. In that set-up, the entire school can be fitted to the needs and purposes of a selected group of students. One might expect that large comprehensive high schools would provide at least a few courses of sufficient rigor to challenge the most able. However, the education of mentally gifted children as a separate group is one of the relatively new programs in public education. Nevertheless, there are a few school systems where groups of children with scientific interests meet with specially qualified teachers who are sympathetic with the idea of developing special talents to a maximum extent, in classrooms equipped to allow the student freedom for individual investigations. It could happen that a comprehensive investigation would challenge some of the materials and methods that long have

been accepted as a matter of course. Thus, the Cooperative Committee raises the question whether the finest algebra and geometry courses of the traditional type are the best mathematics courses that can be provided for the future scientist. In any case, a survey of the type suggested here seems necessary, for certainly neither science nor mathematics teachers have ever come to grips with the design of curricula that are best for youth with special aptitudes for science and mathematics.

**(8).** There is vagueness concerning the content of science and mathematics courses in general education. To be sure there is also confusion concerning the question, "What is the training in art, music, citizenship, and every other field, that must be considered a part of general education?" The high school has, since the turn of the century, been concerned with general education in that it has had the dual responsibility of providing (a) rigorous training for leadership in science, mathematics, and other learned fields, and (b) good general education for better living and for use in the common affairs of life. The rising tide of students has swept this problem into the higher institutions. The problem of general education is now on the desk of the college teacher, and presumably will be in the graduate school day after tomorrow. The present controversy among college teachers of science and mathematics as regards early specialization is only one phase of a problem that will not be solved until we have a definition of general education that is widely accepted. Since the research worker should first of all be an educated citizen concerned about human welfare, general education is of paramount importance. To many it will seem impossible to put anything more into the curriculum buckets that already seem to be overflowing. Perhaps there will be plenty of time to teach all that needs to be taught of general education—once we identify what it is. Certain it is that no good undergraduate program for the education of teachers can be built until the content of general education is defined.

The content of science and mathematics in general education should be determined in a way that will be widely accepted. It may be that we need two definitions of general education—one as an essential for citizenship and the other appropriate for the well-educated person. Be that as it may, the perplexing question is first to identify this content of general education. The task obviously would involve comprehensive investigation; and perhaps a National Commission on the Teaching of Science and Mathematics, if and when created, will make this one of its first projects. It should be noted that a first step has been taken in the Second Report of the Commission on Post-War Plans of the National Council of Teachers of Mathematics. In that document there are identified 29 specific concepts and principles in mathematics that are considered essential in the education of all citizens. The task is more urgent at the college level, for here the great crowd of hungry guests are already sitting at the table, though the cooks have only started the meal. The Cooperative Committee puts the problem in these cautious words: "There is general agree-

ment among college teachers of science that our general courses are not as well designed in terms of general education as they might be, and that better orientation of teachers toward the place of science in general education is of high importance" (p. 113).

**(9).** There is a tendency for the general courses in science and mathematics to become stigmatized. It seems difficult to administer general courses in science and mathematics in a way that will make them respectable and desirable. Here the attitude of the teacher no doubt is the determining factor. Far too often he is by training disposed to give a halo of prestige to the older, traditional, sequential courses. This attitude may be due in part to the fact that many of our high school teachers have never had a single general course either in high school or in college. Then too, teachers do not make clear to students and parents that both types of courses—the general and the specialized—are worthwhile for the pupils who need them, and that they offer different experiences for pupils with very different needs. The general courses have not had a good program of publicity. There is still another factor that tends to stigmatize the general courses. It is the practice, in some schools, of attempting to get homogeneous groups by classifying pupils on the basis of intelligence tests alone.

Teachers of science and mathematics need to recognize the importance of the general courses. Two objectives are involved: (a) a public that will vote intelligently on issues involving the role of science in the world today, and (b) a more abundant and meaningful life for the citizen. Teachers need to reject the fallacy that there is a conflict between general education and the rigorous training of the future scientist. We are not dealing with an either/or proposition. Both tasks must be done and done extremely well. Every department of science and of mathematics should include some teachers who see clearly the function of the general courses.

**(10).** The offering in science and mathematics in the small high school is pathetically meager. Many persons do not realize that more than two-thirds of all high schools are small, with certainly fewer than 200 students and probably fewer than 8 teachers. Such small high schools enroll in all more than a million pupils. In many schools the offering in science and mathematics is limited to the traditional sequential courses designed for college preparation, in spite of the fact that a very small fraction of the graduates ever enter college. On the other hand, some small high schools offer only the general courses in science and mathematics and still others provide no science and mathematics at all in a particular grade.

The Cooperative Committee has made three suggestions. In the first place, the small school can enrich its offerings by teaching simultaneously in small classes of perhaps only a half a dozen students two courses on different levels—one on the level of general education, and one providing a rigorous course for the future science specialist. A second suggestion is to cycle the course in science and mathematics. Physics does not need to be taught every year in order for a

student, with reasonable planning, to pick it up before graduation. Finally, courses in mathematics may be provided for gifted students by correspondence courses that are now serviced by more than 80 higher institutions.

**(11).** Efficiency of instruction in science and mathematics in grades 1–6 is low. Science instruction at all levels leaves much to be desired. The Cooperative Committee includes some distinguished scientists who are no less critical of advanced courses in their own fields than they are of science in the lower grades. However, it is of the greatest importance that the foundation in science instruction be solid. Therefore, the following indictments are cited: (a) Science instruction is in many schools incidental. (b) It too often neglects the natural phenomena that surround the child. (c) The science and mathematics materials are not geared to each other. (d) In arithmetic there is often not sufficient effort to teach meaning—often pupils do not understand what they are doing even when they get the correct answers. (e) The general level of competence in arithmetic is very low (p. 63). (f) Basic concepts and principles are taught by reading and verbalizing, rather than through the varied experiences with the materials in the immediate environment. (g) Symbols and generalizations too often run ahead of experience. (h) In many schools there is no list of specific objectives for the school year (p. 66). (i) The method of science is not in the spotlight. (j) The science and mathematics work of these grades is not planned with the teachers of the next six higher grades to form a 12-year unit of continuous, systematic instruction.

More assistance to classroom teachers must be provided through the use of supervisors, science consultants, and others, who are able to design or recast curriculum materials and to integrate science with other subject areas. School boards might well employ competent teachers of science and mathematics in the summer months to build source units and otherwise organize the curriculum in science and mathematics. Another suggestion is to allow competent teachers time in the school day to plan the units of work. All of these suggestions are important for the reason that the undergraduate curriculum so seldom teaches prospective teachers how to collect and organize instructional materials. It may be that this task requires a degree of maturity and an amount of experience quite beyond the undergraduate. In any event, some teacher-training institutions should provide for at least one year of graduate work in which ample courses in science and mathematics are offered to meet the needs of the teacher instead of the research scientists.

**(12).** The main goal of science and mathematics in the junior high school grades has not been recognized. In both science and mathematics the offerings in these grades have been characterized by great variation. A good illustration is to be found in the case of mathematics for the ninth school year. In adjacent towns one may find the following offerings in small high schools: School *A* offers only general mathematics; school *B* provides only algebra; school *C* teaches a diluted algebra; school *D* requires pupils to elect commercial arithmetic; school

*E* teaches agriculture mathematics only; and school *F* teaches no mathematics at all in order to cut down failures! While this illustration is a bit extreme, the picture of science offered in grades 7 and 8 would also be shocking.

Grades 7, 8, and 9 provide the crucial years for achieving the objectives of general education. Most of the youngsters at this age level are still in school, where they can be taught those things which society deems essential. The general courses have greater flexibility, and therefore may include a greater variety of activities to meet the needs of varied interests and abilities. The foregoing statement does not imply that all ninth grade students should be enrolled in general mathematics and general science. A goodly fraction of the students will have achieved functional competence in mathematics by the end of the eighth school year. Moreover, the future scientist is almost certain, by that time, to want more organization, more rigor, more continuity than can be or should be provided in the general courses. In brief, the beginning of the ninth year is the time for differentiation of courses and classification of students. In the ninth grade we should operate a double track, at least, in mathematics and in science.

**(13).** As regards senior high school, no one should assume that all is well with the traditional courses in science and mathematics. In most schools they are woefully out of date as regards both content and method. The inertia is largely due, as has been pointed out earlier, to the fact that teachers are not given time in the school day or employed in the summer months to do the technical and arduous task of curriculum revision.

Science and mathematics teachers should check the content and method of the traditional sequential courses with the recommendations that have long been advocated in the reports of our national committees. We will do well to follow the example of the industrialists and meet new needs with improved materials, and with more efficient methods. We should recognize that future scientists are not desirious that we make these courses easy. Moreover, the traditional sequential courses should be reserved for the pupils who can profit by such courses. The unique values of the traditional courses can not be achieved by a constant gearing down. Failure should, for the most part, be avoided by guiding pupils to appropriate general courses.

**(14).** The science and mathematics courses in the junior college have been allowed to develop without much design. These courses have grown up like Topsy, and, as concerns the sequential courses, largely in imitation of freshman and sophomore courses in the colleges. On the other hand, an inspection of the great variety of new general courses suggests that a systematic evaluation might uncover many courses that are superficial and lacking in specific objectives. The junior college, or perhaps better, the community college, is now undergoing its second great spurt of growth. Witness the fact that more than one-fourth of the institutions of higher learning are junior colleges. In the fall of 1946 there were 119,000 new students enrolled in junior colleges as compared to 163,000 new students in colleges of arts and sciences, and 269,000 in universities and large

institutions of complex organization (p. 111). The evidence that the community college is here to stay and that it will play an important part in public education is completely convincing. It is clear that national organizations can not afford to neglect the science and and mathematics of so large a group of college students.

We need to provide science and mathematics courses for three very different types of students enrolled in the community college. It is assumed that the community college provides educational opportunities which otherwise might be inaccessible to a large number of educatable youth. In that case the three types of courses called for are: (a) courses that attempt to guarantee youngsters competence in mathematics and science as regards the content of general education for all who can possibly achieve it, (b) courses that provide certain science and mathematics competencies needed by students who have a desire to follow specific vocational interests, and (c) rigorous sequential courses in science and mathematics for students who plan a career in science or mathematics. As a minimum it would seem that the community college should offer (1) at least a year of mathematics and of science which is general in appeal, flexible in purpose, challenging in content, and functional in outcome, (2) a one-year pre-vocational course consisting of units that correlate materials from mathematics, physics, and industrial arts, and (3) ample provision for the student with a major interest in science and mathematics.

**(15).** Too often college courses are geared to the needs of the future specialist, to the neglect of the general student body whose main interest is in the science and mathematics of general education. This, in the judgment of the Cooperative Committee, is one of the main weaknesses of the science and mathematics programs in the colleges. This criticism seems valid also for many other undergraduate courses—even in the social studies, as, for example, sociology, psychology, and economics. Perhaps the freshman courses are the worst offenders, although there have been some sincere efforts, such as those at Columbia, Chicago, and Harvard, to design courses that would provide the science and mathematics of general education for the educated person. However, the typical freshman course in mathematics, when evaluated in terms of the needs of women, for example, suggests that we still have a long way to go.

A department of science and mathematics in a college needs to give careful attention to its functions as a service unit to other departments—medicine, economics, etc. Quoting directly from the report:

> In many of the large State universities and institutions of complex organization, the science departments are largely service departments for the professional schools. In many cases it is not the ability of the student but his membership in a professional group which determines the type of science offering the student has to take. The insistence of many professions that every student, during his college career, be exposed to every branch of the natural sciences, produces an antiscientific attitude on the part of the student who does not understand basic principles but is forced to assimilate superficially an agglomeration of facts and formulas. It produces a frustration on the part of the science teacher who is unable to hold and inspire the majority of his students (p. 118).

In an earlier section, the reader's attention was called to the controversy among college teachers of science and mathematics as regards early specialization. Perhaps a precise identification of the elements of science and mathematics in general education will go a long way in resolving this difficulty. In the meantime, the able and interested student should have the opportunity of starting courses in his special field as early as he knows what his field is, and certainly by the beginning of the second year of college (p. 123). A drastic indictment of science and mathematics in the colleges is that "In a great many institutions departmentalized instruction is so rigorous that scientific progress is impeded by it" (p. 127). To solve this problem, the Cooperative Committee suggests that "Advances can be expected at the borderline of the sciences and special attention should be given to the creation of strong departments in universities to cover overlapping fields of science such as meteorology, geophysics, geochemistry, biophysics, radiobiology, and to develop these newly recognized fundamental areas. These borderline fields have been developed by capable individuals, partly self-trained, but they merit the attention of the curriculum planners and justify specially trained scientific personnel for further fundamental work in these areas" (p. 127).

**(16).** The rapid expansion of enrollments in graduate work is creating serious problems. The discussion here will treat lightly such aspects of the situation as inadequate material and lack of space. There are certain problems stemming from heavy enrollments that will not be solved easily and probably not very soon. Such problems may be illustrated by the following: (1) With fewer contacts, desirable student-professor relationships are now more difficult to build. (2) There is a lack of competent supervision in the classroom and laboratory. (3) It is even more difficult than in the past to give teaching the full attention which it deserves by placing the most experienced, capable, and inspiring members of the staff in charge of the all-important task of introducing the beginning students to science (p. 117). (4) The value of the master's degree is being deflated. In this paper we shall discuss only the problems concerning the master's degree and the need for supervision.

The vast up-surge in enrollment makes the definition of the master's degree one of the major problems of the graduate school. There are two attacks on the master's degree, and curiously enough, they come from opposite directions. In the first place, some scientists insist that the master's degree is no longer a good screening device for selecting the competent research worker. This is due not only to lack of uniformity in requirements for the master's degree, but to gearing down of standards under the pressure of large enrollments. The other criticism is expressed by those who recognize that in many institutions the master's degree has for some time been, to no small extent, a teacher's degree. This does not imply that the curriculum in such institutions has been especially designed to serve the needs of prospective teachers, but rather that a goodly fraction of the graduates, if they continued their education, turn out to be teachers, espe-

cially in the lower schools. As regards the master's degree, the Cooperative Committee recommends: A master's degree designed as a powerful screening device for the doctoral candidate that would require evidence of research ability, the competence to find and evaluate the literature bearing on a given piece of research, the discerning mind to choose significant problems of investigation, and the ability to devise effective means of solving such problems. It is hoped that if graduate schools define the degree of master of science as proposed in this report, that a truly professional master's degree will also be made available to teachers. Indeed it might be feasible and desirable to challenge a teacher to strive for two master's degrees: (a) A master's degree (M.A. in Education) given upon the completion of a curriculum that represents a conscientious effort to meet the needs of future teachers, especially in elementary and secondary schools. Under sensible administration this degree need not turn out to be a stigmatized "cheap" degree, for in that case the teachers would be selected for high competence on the total college record which obviously would include many courses in which future research workers would also be enrolled. (b) A master's degree, perhaps called Master's Degree in Professional Education, to be awarded to educators who have completed two years of graduate work, and who have convincingly demonstrated that they are master teachers or highly skilled practitioners in other phases of professional education. The argument is that sooner or later we must differentiate between the competent teacher of science and mathematics and the mere transient and the utterly incompetent.

**(17).** One of the low spots is the training now offered in statistical methods. The Committee on Applied Mathematical Statistics* reports that: "There has been an unprecedented growth of interest in the use of statistical methods during the past 10 years, which has caught the American educational system unprepared. . . . There is a shortage of adequately trained statisticians in both academic and non-academic categories and a more severe shortage is expected. . . . The shortage of training facilities for mathematical and especially for applied statisticians is critical. . . . "

Mathematics departments of colleges should provide the kind of statistical training as a service to other departments that will bridge the gap between theory and applications. Operating as a service unit, the mathematics department of a college should provide statistical training needed for students whose major interest is in other fields. The problem will not be solved by requiring students in these various fields to take one or two courses of theoretical statistics. Rather the department of mathematics needs to provide types of statistical training that will bridge the gap between theory and the applications in these fields. Undergraduate courses even in the social studies are becoming more systematic and therefore more statistical. Undergraduates to an increasing de-

* "Personnel and Training Problems Created by the Recent Growth of Applied Statistics in the United States," a report by the Committee on Applied Mathematical Statistics, National Research Council, Reprint and Circular Series, No. 128, May 1947. For a summary of this report, refer to this MONTHLY, vol. 54, 1947, p. 525.

gree are not prepared to deal with the statistical problems that they encounter. Neither the college nor the high school is doing a good job in meeting this need. Some one should analyze the statistics needed in the undergraduate courses. In all probability a full semester of high school training in statistics—an elective—would be indicated. If this should turn out to be the case, colleges might well encourage the strong high schools to offer a semester in statistics. Finally, colleges and universities should keep an eye on the rapid development in statistics and should realize that the demand for persons well trained in statistics is likely to exceed the supply in the years immediately ahead.

**(18).** It is inefficient and uneconomical for a teacher of science to spend an excessive amount of time as a technician or as a serviceman for his own apparatus. It is generally recognized that teachers in all subjects waste precious time doing clerical tasks that should go into instruction. However, the fact that time and effort is wasted by science instructors on repairing and rebuilding apparatus is not realized by most administrators. Moreover, in high school courses these tasks should be done by selected students with a bit of special training, and by hired technicians at the college and university level, for the reason that in the learning stage such work may provide valuable experiences. Incidentally, our schools probably do not give students enough of the practice in leadership they would get by appearing before a class or a group with responsibility for such specific tasks.

In high school, college and university, we need to utilize trained assistants and hired technicians to increase the efficiency of science teachers. Prospective teachers of the high school, if they are going to teach students the skills of a technician, must themselves learn to repair, design and rebuild pieces of equipment and this means they must be able to use glass blowing techniques, hand tools, and some machine tools. Quoting from the report:

> It is especially important under the conditions now existing in graduate schools that adequate technician services be available. Otherwise those who are directing the research of graduate students will be immersed in a mass of detail which will materially reduce the number of students whom they can supervise. In addition to machinists, and glass blowers for the physical sciences, and technical assistants for the biological sciences, both now require electronics aids of the type which did much of the development and all of the servicing of radar and communications equipment during the war. The new electrical techniques are as important in the physiology laboratory as in the physics laboratory. Because they are new, the necessity for that type of technician is not yet recognized by many university administrators, who will have to realize their importance (p. 138).

**(19).** The scientific approach to problems is not given enough emphasis. It is reasonable to assume that the scientific method would be about the last thing to be neglected in courses in science. This, however, does not seem to be true. Throughout the report, whether the discussion deals with science in the early grades of the elementary school, or with the advanced courses in the college, there is the implied criticism that the main objective of science instruction is not being achieved to the extent desired. Let us consider this sample quotation: "The scientific approach to problems of the nature of man should occupy a cen-

tral role both in the courses offered in general education and those offered to science specialists. In the former, insufficient stress is now laid on the objectivity of scientific approach and the method of scientific analysis. An accumulation of facts, no matter how well stated or numbered, fails to provide the student with the tools he so urgently needs, namely those of searching inquiry, critical analysis, and unbiased conclusions" (p. 114).

In the science courses at all levels, both in general education and in the training of the future scientist, there should be an emphasis on the method of science. It is understandable that there might be a neglect of the techniques of scientific method in general courses in science and mathematics. However, the following quotation implies that the scientific method is not even given adequate attention in the traditional sequential courses. "In courses for science specialists the scientific method usually appears by implication and in time some students acquire it as second nature. But frequently a conscious and explicit exposition of the scientific approach and its implication is lacking. More effort should be spent in clarifying the nature of scientific thought and its historic development and the cooperative aspects of investigations, the free nature of exchange of ideas and findings, and the important effects of scientific discoveries on our society" (p. 114). Finally it is recommended that courses be given in which a conscious effort is made to clarify the nature of scientific thought, the cooperative aspects of scientific investigations, the ethical implications of the free exchange of ideas and the important effects of scientific discoveries on society. We also recommend that courses be required in the history of scientific thought, in the philosophy of science and in general philosophy, so that the science specialist may recognize the social implications of scientific endeavor (p. 143). The growth of interest in a college course on the Foundations of Mathematics is to be commended.

**(20).** As regards efficiency of science instruction, the ceiling is low, for we do not know the answer to many of our problems in science education. A host of systematic studies are needed to provide the data from which sensible solutions may evolve. These studies, now so urgently needed, require funds and personnel far greater than any existing organization can provide. The task is beyond what any higher institution or any single national society can do.

Some national organization should be set up to study the problems in science education in a systematic, a comprehensive, and a reliable manner. There are at least two possibilities. In the first place there is the hope that the National Science Foundation, if and when created, will make available the funds and the personnel to undertake the necessary investigations. In the second place, the Cooperative Committee is recommending that a National Commission on the Teaching of Science and Mathematics be created. Some form of cooperative action on matters relating to science education by the national societies is clearly indicated, and almost certain to prove useful. In the event that the Science Research Foundation is not established, cooperative action would seem to be a necessity.

## MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

*Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.*

### RATIONAL SOLUTIONS OF A DIOPHANTINE EQUATION

P. A. CARIS, University of Pennsylvania

**1. Introduction.** It is the purpose of this paper to find a general non-trivial rational solution of the diophantine equation

$$u^3 + v^3 + w^3 - 3uvw = x^3 + y^3. \tag{1}$$

The result, though readily obtained, is believed to be new.

First it may be remarked that if $u+v+w=0$, or if $u=v=w$, then $x+y=0$. Conversely, if $x+y=0$, either $u+v+w=0$, or $u=v=w$. The latter result comes from the fact that the quadratic factor of $u^3+v^3+w^3-3uvw$ is a definite form. Such solutions are trivial and will be ignored in this discussion.

The substitutions, $u=r-s$, $v=r+s$, $w=r+t$, transform (1) into

$$(3r + t)(t^2 + 3s^2) = x^3 + y^3.$$

If $t=s=0$, then $u=v=w$. Hence it will be assumed that $ts\neq 0$.

**2. Solution.** Multiply the above transformed equation by 4 for the sake of convenience. Then (1) may be written thus:

$$4(3r + t)(t^2 + 3s^2) = (x + y)[(2x - y)^2 + 3y^2].$$

This equation will be satisfied in rational numbers if the following system of equations is satisfied in rational numbers:

$$\begin{aligned} p(3r + t) &= q(x + y), \\ 2q(t + s\sqrt{-3}) &= (a + b\sqrt{-3})(2x - y + y\sqrt{-3}), \\ 2(a + b\sqrt{-3})(t - s\sqrt{-3}) &= p(2x - y - y\sqrt{-3}). \end{aligned}$$

This system will be satisfied if

$$3pr + pt = qx + qy, \tag{2}$$

$$2qt = 2ax - ay - 3by, \tag{3}$$

$$2qs = ay + 2bx - by, \tag{4}$$

$$2at + 6bs = 2px - py, \tag{5}$$

$$py = 2as - 2bt. \tag{6}$$

Here $x$, $y$, $r$, $s$, $t$ are to be found. Assume however, for the moment, that these values are known, i.e., assume that $(3r+t)(t^2+3s^2)=x^3+y^3$. Then the following values,

$$p = 2t^2 + 6s^2,$$

$$q = 2(x^2 - xy + y^2),$$
$$a = 2tx - ty + 3sy,$$
$$b = 2sx - ty - sy,$$

satisfy the system of equations, as can readily be seen by substitution. Hence, to find the general solution of (1), it will suffice to solve the system of equations for $x, y, r, s, t$.

Now write equation (3) thus:

$$2pqt = a(2px - py) - 3b(py).$$

Substitute in this equation from (5) and (6). Then

$$2pqt = 2a^2t + 6b^2t,$$

whence, since $t \neq 0$, $pq = a^2 + 3b^2$.

Similarly, (4) will be satisfied by (5) and (6) with $pq = a^2 + 3b^2$, since $s \neq 0$.

From (5), making use of (6), we get

$$px = (a - b)t + (a + 3b)s.$$

Now write (2) thus:

$$3p^2r + p^2t = q(px + py).$$

In this equation substitute the values of $px$ and $py$ already known. Then

$$3p^2r + p^2t = q(a - 3b)t + q(3a + 3b)s,$$

whence

$$3p^2r = (aq - 3bq - p^2)t + q(3a + 3b)s.$$

Here let $t = 3cp^2$ and $s = dp^2$. Since $t \neq 0$, we have $p \neq 0$. Then

$$r = acq - 3bcq - cp^2 + adq + bdq.$$

These values for $t$ and $s$, when substituted in the equation found for $px$, give

$$x = 3acp - 3bcp + adp + 3bdp,$$

and, similarly, from (6),

$$y = 2adp - 6bcp.$$

Hence,

$$u = r - s = acq - 3bcq - cp^2 - dp^2 + adq + bdq,$$
$$v = r + s = acq - 3bcq - cp^2 + dp^2 + adq + bdq,$$
$$w = r + t = acq - 3bcq + 2cp^2 + adq + bdq,$$
$$x = 3acp - 3bcp + adp + 3bdp,$$
$$y = 2adp - 6bcp,$$

where $pq=a^2+3b^2$.

As a special case, the values $a=1$, $b=-1$, $p=q=2$, give the neat identity:

$$(c-d)^3+(c+d)^3+(4c)^3-12c(c^2-d^2)=(3c-d)^3+(3c+d)^3.$$

## ON A CONTACT TRANSFORMATION THEOREM

HUAN-TING KUO, National Wuhan University, Wuchang, China

The aim of this paper is to give a vectorial proof of the following classic contact transformation theorem.

THEOREM. *Given a surface $S$ and a fixed point $O$; join the point $O$ to any point $M$ of the surface $S$, and pass a plane $OMN$ through $OM$ and the normal $MN$ to the surface $S$ at the point $M$. In the plane $OMN$ draw through the point $O$ a perpendicular to the line $OM$, and lay off on it a length $OP=OM$. The point $P$ describes a surface $\Sigma$, which is called the apsidal surface to the given surface $S$. The transformation is a contact transformation, and the relation between the surfaces $S$ and $\Sigma$ is a reciprocal one. When the given surface $S$ is an ellipsoid and the point $O$ is its center, the surface $\Sigma$ is Fresnel's wave surface.*

*Proof.* Take $O$ as the origin of a system of rectangular coordinates. Let the coordinates of $M$ and $P$ be $x, y, z$ and $X, Y, Z$ respectively. Then obviously we have

$$xX+yY+zZ=0, \tag{1}$$

$$X^2+Y^2+Z^2=x^2+y^2+z^2, \tag{2}$$

$$\begin{vmatrix} x & y & z \\ X & Y & Z \\ p & q & -1 \end{vmatrix}=0. \tag{3}$$

Differentiating (1) and (2), transposing, and dividing by $\sqrt{x^2+y^2+z^2}$ or $\sqrt{X^2+Y^2+Z^2}$, we get

$$\frac{xdX+ydY+zdZ}{\sqrt{x^2+y^2+z^2}}=-\frac{Xdx+Ydy+Zdz}{\sqrt{X^2+Y^2+Z^2}}, \tag{4}$$

$$\frac{XdX+YdY+ZdZ}{\sqrt{X^2+Y^2+Z^2}}=\frac{xdx+ydy+zdz}{\sqrt{x^2+y^2+z^2}}. \tag{5}$$

Let us denote the vectors $(dx, dy, dz)$ and $(dX, dY, dZ)$ by $a$ and $A$ respectively. Using $OM$ and $OP$ as the first and the second axes of a rectangular system, and denoting the components of $a$ and $A$ referred to this system of coordinates by $a_1, a_2, a_3$ and $A_1, A_2, A_3$ respectively, we have from (4) and (5)

$$A_1=-a_2, \qquad A_2=a_1.$$

Thus

$$a_1A_1 + a_2A_2 = -a_1a_2 + a_2a_1 = 0.$$

Consequently the projection of $(dx, dy, dz)$ on the plane $MOP$ is perpendicular to the projection of $(dX, dY, dZ)$ on this same plane. But the vector $(dx, dy, dz)$ always lies on the tangent plane at $M$, hence the projection of $(dx, dy, dz)$ on the plane $MOP$ is along the line of intersection of $MOP$ and the tangent at $M$. It follows that the tangent plane at $P$ is perpendicular to the tangent plane at $M$. Thus the transformation is a contact transformation. Obviously the normal at $P$ is on the plane $MOP$, and we have

$$\begin{vmatrix} x & y & z \\ X & Y & Z \\ P & Q & -1 \end{vmatrix} = 0. \tag{6}$$

By (1), (2), (3) and (6) we know the transformation is a reciprocal one. To find $P$ and $Q$ we use the equation $pP+qQ+1=0$ and the equation (6).

---

## CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

*All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania.*

### GENERALIZATION, SPECIALIZATION, ANALOGY*

GEORGE PÓLYA, Stanford University

My personal opinion is that the choice of problems and their discussion in class must be, first and foremost, *instructive*. I shall be in a better position to explain the meaning of the word "instructive" after an example. I take as an example the proof of the best known theorem of elementary geometry, the theorem of Pythagoras. The proof on which I shall comment is not new; it is due to Euclid himself (*Elements* VI, 31).

1. We consider a right triangle with sides $a$, $b$ and $c$, of which the first, $a$, is the hypotenuse. We wish to show that

$$a^2 = b^2 + c^2. \tag{1}$$

This aim suggests that we describe squares on the three sides of our right triangle. And so we arrive at the not unfamiliar part I of our compound figure.

* Presented at the summer meeting of the Mathematical Association of America, New Haven, Conn., September 1, 1947.

(The reader should draw the parts of this figure as they arise, in order to see it in the making.)

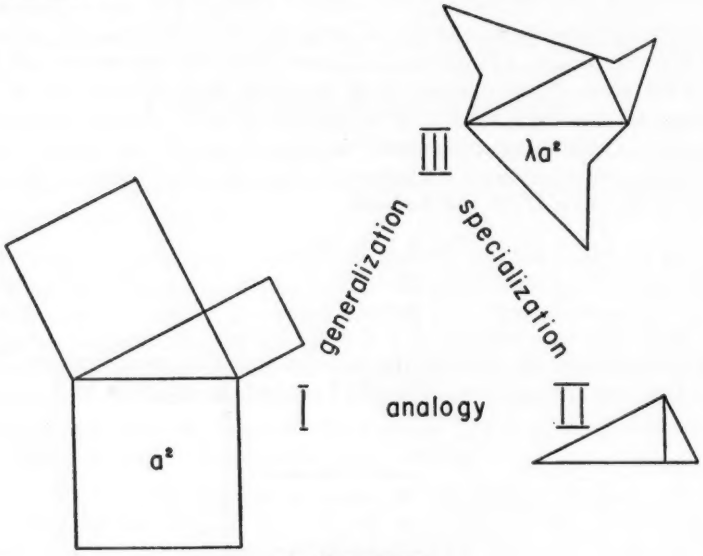


Fig. I, II and III

2. Discoveries, even very modest discoveries, need some remark, the recognition of some relation. We can discover the following proof by observing the *analogy* between the familiar part I of our compound figure and the scarcely less familiar part II: the same right triangle that arises in I is divided in II into two parts by the altitude perpendicular to the hypotenuse.

3. Perhaps, you fail to perceive the analogy between Figures I and II. This analogy, however, can be made quite explicit by a common *generalization* of I and II which is expressed by Figure III. There we find again the same right triangle, and on its three sides three polygons are described which are similar to each other but arbitrary otherwise.

4. The area of the square described on the hypotenuse in Figure I is $a^2$. The area of the irregular polygon described on the hypotenuse in Figure III can be put equal to $\lambda a^2$; the factor $\lambda$ is determined as the ratio of two given areas. Yet then, it follows from the similarity of the three polygons described on the sides $a$, $b$ and $c$ of the triangle in Figure III that their areas are equal to $\lambda a^2$, $\lambda b^2$ and $\lambda c^2$, respectively.

Now, if the equation (1) should be true (as stated by the theorem that we wish to prove), then also the following would be true:

$$\lambda a^2 = \lambda b^2 + \lambda c^2. \tag{2}$$

In fact, very little algebra is needed to derive (2) from (1). Now, (2) represents a *generalization* of the original theorem of Pythagoras: *If three similar polygons are described on the three sides of a right triangle, the one described on the hypotenuse is equal in area to the sum of the two others.*

It is instructive to observe that this generalization is *equivalent* to the special case from which we started. In fact, we can derive the equations (1) and (2) from each other, by multiplying or dividing by $\lambda$ (which is, as the ratio of two areas, different from 0).

5. The general theorem expressed by (2) is equivalent not only to the special case (1), but to any other special case. Therefore, if any such special case should turn out to be obvious, the general case would be demonstrated.

Now, trying to *specialize* usefully, we look around for a suitable special case. Indeed Figure II represents such a case. In fact, the right triangle described on its own hypotenuse is similar to the two other triangles described on the two legs, as is well known and easy to see. And, obviously, the area of the whole triangle is equal to the sum of its two parts. And so, the theorem of Pythagoras has been proved.

6. I took the liberty of presenting the foregoing reasoning so broadly because, in almost all its phases, it is so eminently instructive. A case is instructive if we can learn from its something applicable to other cases, and the more instructive the wider the range of possible applications. Now, from the foregoing example we can learn the use of such fundamental mental operations as generalization, specialization and the perception of analogies. There is perhaps no discovery either in elementary or in advanced mathematics or, for that matter, in any other subject that could do without these operations, especially without analogy.

The foregoing example shows how we can ascend by generalization from a special case, as from the one represented by Figure I, to a more general situation as to that of Figure III, and redescend hence by specialization to an analogous case, as to that of Figure II. It shows also the fact, so usual in mathematics and still so surprising to the beginner, or to the philosopher who takes himself for advanced, that the general case can be logically equivalent to a special case. Our example shows, naively and suggestively, how generalization, specialization and analogy are naturally combined in the effort to attain the desired solution. Observe that only a minimum of preliminary knowledge is needed to understand fully the foregoing reasoning. And then we can really regret that mathematics teachers usually do not emphasize such things and neglect such excellent opportunities to teach their students to think.*

---

* The author's views are presented more fully in his booklet, *How to Solve It* (Princeton, 5th enlarged printing 1948). For more about generalization, specialization and analogy see the sections starting on pp. 97, 164 and 37.

## AREAS OF PLANE FIGURES

HOI-CHEUNG PANG, Provincial Canton Technical High School, China

**1. Triangles.** A well known formula of analytic geometry states that the area of a triangle whose vertices are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ is given by:

$$(1) \qquad \text{Area } (P_1P_2P_3P_1) = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3).$$

If the passage around the triangle places the interior of the triangle on the left, (1) gives a positive result; the opposite direction of passage gives a negative result.

This area may also be computed by the following rule:

(a) Write down the vertices in two rows in the order $P_1P_2P_3P_1$

$$\begin{matrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{matrix}$$

(b) Multiply each abscissa by the ordinate of the following column and add, giving: $x_1y_2+x_2y_3+x_3y_1$.

(c) Multiply each ordinate by the abscissa of the following column and add, giving: $x_2y_1+x_3y_2+x_1y_2$.

(d) Subtract the result of (c) from that of (b) and divide by 2. This gives the result in formula (1).

**2. Other polygons.** This rule equally well gives the areas of polygons of all descriptions. The proof consists of dividing the polygon into triangles, and applying the above rule to each triangle dropping out terms which cancel. As an illustration consider quadrilateral $P_1P_2P_3P_4P_1$. Divide the quadrilateral into

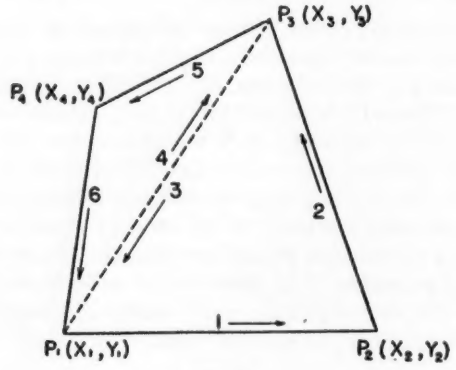


Fig. 1

the triangles $P_1P_2P_3P_1$ and $P_1P_3P_4P_1$ and write down the coordinates of the two triangles in two sets:

$$\begin{array}{cccc} P_1P_2P_3P_1 & & P_1P_3P_4P_1 & \\ x_1 \quad x_2 \quad x_3 \quad x_1 & & x_1 \quad x_3 \quad x_4 \quad x_1 & \\ y_1 \quad y_2 \quad y_3 \quad y_1 & & y_1 \quad y_3 \quad y_4 \quad y_1 & \end{array}$$

Applying the above rule to each triangle we see that the last part of the result obtained from the first triangle, namely $x_3y_1 - x_1y_3$ cancels with the first part of the result obtained from the second triangle, namely $y_1y_3 - x_3y_1$. So we can combine the two sets of coordinates into a third one, namely:

$$\begin{array}{ccccc} P_1P_2P_3P_4P_1 & & & & \\ x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_1 & & & & \\ y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_1 & & & & \end{array}$$

and apply the above rule to find the area of the quadrilateral with the following caution in step (a):

(a′) Write down the coordinates of the vertices in an order agreeing with that established by passing continuously around the perimeter.

This proof generalizes at once to any non-self-intersecting polygon—whether convex or not. When the polygon intersects itself the procedure gives a result whose interpretation is suggested by the example:

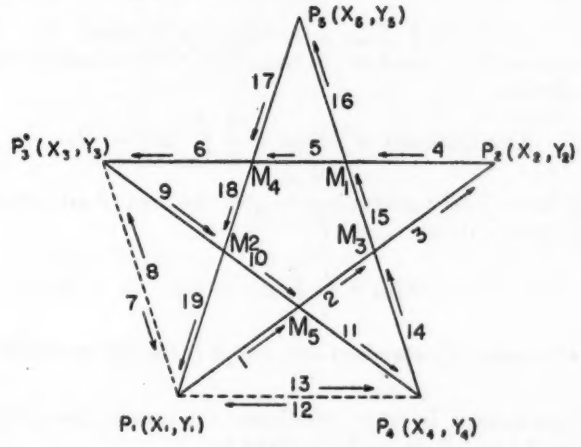


Fig. 2

Following the arrows around the perimeter we note that 7 and 8 and also 12 and 13 cancel with each other. The area given by the rule represents the sums of the areas of:

$$\Delta P_1M_5M_2P_1 + \Delta P_4M_3M_5P_4 + \Delta P_2M_1M_3P_2 + \Delta P_5M_4M_1P_5 + \Delta P_3M_2M_4P_3 + 2(\text{Pentagon } M_1M_4M_2M_5M_3M_1).$$

**3. Closed curves.** Through a limiting process the rule gives a formula for the area bounded by a closed curve. For example divide the arc of the first quadrant

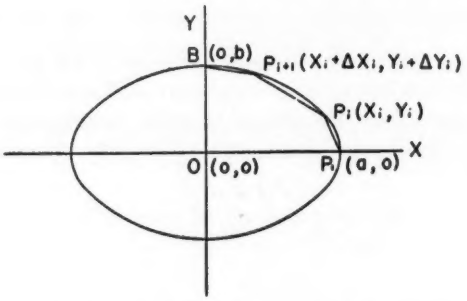


Fig. 3

of an ellipse into $n$ divisions, equal or unequal. Let $P_i$ and $P_{i+1}$ be two consecutive points of division, and apply the rule to the polygon $P_1 \cdots P_iP_{i+1} \cdots BOP_1$. We have the array:

$$x_1(=a) \cdots x_i \quad x_i + \Delta x_i(=x_{i+1}) \cdots x_{n+1}(=0) \quad 0 \quad a$$
$$y_1(=0) \cdots y_i \quad y_i + \Delta y_i(=y_{i+1}) \cdots y_{n+1}(=b) \quad 0 \quad 0$$

So Area $P_1 \cdots P_nBOP_1 = \frac{1}{2} \sum_{i=1}^{n} x_i(y_i+\Delta y_i) - y_i(x_i+\Delta x_i)$. The limit of this area gives the area of the quadrant of the ellipse. Hence from the fundamental theorem of calculus:

$$\text{Area Quadrant of Ellipse} = \frac{1}{2}\int_{P1}^{B} x\,dy - y\,dx. \tag{2}$$

This formula immediately generalizes to give the area of any closed curve as a line integral around its boundary, $C$:

$$\text{Area} = \frac{1}{2}\int_{C} x\,dy - y\,dx. \tag{3}$$

This result, of course, is a standard one, but it is usually proved in a different fashion.

**4. Polar coördinates.** In polar coördinates the area of the triangle $OP_1P_2O$ where $P_1$ is $(\rho_1, \theta_1)$ and $P_2$ is $(\rho_2, \theta_2)$ is given by:

$$\text{Area } OP_1P_2O = \tfrac{1}{2}\rho_1\rho_2 \sin(\theta_2 - \theta_1). \tag{4}$$

Hence the area of any polygon of $n$ sides is

$$\text{Area } P_1P_2 \cdots P_nP_1 = \frac{1}{2}\sum_{i=1}^{n} \rho_i\rho_{i+1} \sin(\theta_{i+1} - \theta_i) \tag{5}$$

assuming $(\rho_{n+1}, \theta_{n+1})$ is $(\rho_1, \theta_1)$.

Passing to the case of a curve, we have that the area between radii whose polar angles are $\alpha$ and $\beta$ is given by

$$\text{Area} = \frac{1}{2}\lim_{n\to\infty}\sum_{i=1}^{n}[\rho_i(\rho_i+\Delta\rho_i)\sin\Delta\theta_i]$$

$$= \frac{1}{2}\lim_{n\to\infty}\sum_{i=1}^{n}\rho_i\Delta\theta_i + \frac{1}{2}\lim_{n\to\infty}\sum_{i=1}^{n}\rho_i\Delta\rho_i\Delta\theta_i$$

$$= \frac{1}{2}\int_{\alpha}^{\beta}\rho^2 d\theta. \tag{6}$$

For $\lim_{x\to 0}(\sin x/x)=1$ and $\Delta\rho_i\Delta\theta_i$ is an infinitesimal of higher order than the first.

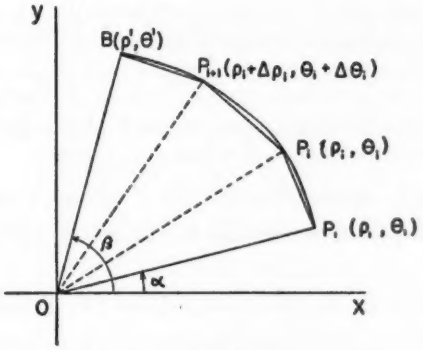


Fig. 4

## CORRECTION

In the article, *A Problem of Collinear Points*, by H. S. M. Coxeter, this MONTHLY, vol. 55, p. 27 the first displayed formula should read:

$$x^3 + y^3 + z^3 = xyz = 0.$$

## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

### PROBLEMS FOR SOLUTION

E 811. *Proposed by H. D. Larsen, Albion College, Michigan*

$A$, $B$, and $C$ participate in a novel dart game, the targets consisting of three small balloons marked $A$, $B$, and $C$, respectively. At each turn one dart is thrown, the order of the turns being determined in advance by drawing lots. As soon as a balloon is hit and destroyed, the owner of that balloon is eliminated from the game. The balloons are placed in such a manner that there is no danger of destroying a balloon by a dart aimed at another balloon. It is known by all participants that $A$ can hit a balloon 4 out of 5 times, $B$ 3 out of 5 times, and $C$ 2 out of 5 times; this knowledge is used by each player to his best advantage. What is each contestant's chance of winning the game?

E 812. *Proposed by Monte Dernham, San Francisco, California*

Find the shortest perimeter common to two different primitive Pythagorean triangles.

E 813. *Proposed by C. W. Trigg, Los Angeles City College, California*

Let $S$ be the sum of the integer elements of a magic square of order three, and let $D$ be the value of the square considered as a determinant. Show that $D/S$ is an integer.

E 814. *Proposed by Sidney Kravitz, New York, New York*

Given the curve $y=e^x/x$. Consider all areas under the curve, over the $x$-axis, and between two ordinates one unit apart. Locate the boundary lines of the area which is a minimum.

E 815. *Proposed by J. S. Miller, Dillard University*

A marble rolls in a hemispherical bowl. Find its period.

### SOLUTIONS

#### Another Ball in a Vase Problem

E 781 [1947, 412]. *Proposed by P. D. Thomas, Navy Department, Washington, D.C.*

A heavy ball is gently dropped into a vase full of water. A section through the vertical axis of the vase is a semiellipse, the height being the semimajor axis, the diameter being then the minor axis. The size of the ball is such as to cause

the maximum displacement. (1) Find the radius of the ball. (2) Show that the plane of the circle of tangency bisects the height of the submerged segment of the ball. (See E 687 [1946, 334].)

*Solution by the Proposer.* Let the semimajor and semiminor axes of the semi-ellipse be $a$ and $b$ respectively. Let the major and minor axes be the $x$ and $y$ axes, and $C$ the origin. Then a point $P$ on the ellipse has coordinates

$$P: \quad (x, b\sqrt{1 - x^2/a^2}).$$

The normal at $P$ meets the $x$-axis in the point

$$G: \quad \left(\frac{a^2 - b^2}{a^2}x, 0\right).$$

The radius $R$ of the variable ball is equal to $GP$; that is

$$GP = R = b\sqrt{1 + (b^2 - a^2)x^2/a^4}. \tag{1}$$

The height $h$ of the segment of the ball submerged is $CG+R$, or

$$h = CG + R = (a^2 - b^2)x/a^2 + R, \tag{2}$$

where $R$ is given by (1). The volume of a spherical segment is

$$V = \pi h^2(3R - h)/3,$$

whence

$$dV/dx = \pi h[(2R - h)dh/dx + hdR/dx]. \tag{3}$$

The second factor of (3), with the values of $R$, $h$, $dR/dx$, $dh/dx$ obtained from (1) and (2), becomes

$$a^2b\sqrt{1 + (b^2 - a^2)x^2/a^4} = (a^2 + b^2)x,$$

whence we find

$$x_{\max} = ab/(a^2 + 3b^2)^{1/2},$$

and therefore from (1)

$$R_{\max} = b(a^2 + b^2)/a(a^2 + 3b^2)^{1/2}.$$

Computing $h$ from (2) we find

$$h = 2ab/(a^2 + 3b^2)^{1/2} = 2x_{\max},$$

which shows that the plane of the circle of tangency bisects the height $h$ of the submerged spherical segment.

Also solved by Ragnar Dybvik.

**Numbers of the Form $x^2+kxy+y^2$**

E 782 [1947, 412]. *Proposed by Joseph Rosenbaum, The Milford School, Connecticut*

Show that the product of two numbers each of the form $x^2+kxy+y^2$, where $k$, $x$, $y$ are integers, $k$ fixed, is also of that form. If $x$ and $y$ are relatively prime, are all factors of $x^2+kxy+y^2$ also of the same form?

*Solution by Leo Moser, University of Manitoba.* The first part is verified by the identity

$$(a^2 + kab + b^2)(c^2 + kcd + d^2) = x^2 + kxy + y^2,$$

where

$$x = ac - bd, \qquad y = ad + bc + kbd.$$

The question is answered in the negative by taking $x=4$, $k=4$, $y=5$. Then

$$x^2 + kxy + y^2 = 121.$$

But we cannot have

$$a^2 + 4ab + b^2 = 11$$

inasmuch as

$$a^2 \equiv 0 \quad \text{or} \quad 1 \pmod 4,$$

and thus

$$a^2 + 4ab + b^2 \equiv 0, 1, \quad \text{or} \quad 2 \pmod 4.$$

Also solved, in part, by the proposer.

**Enumeration of Triangles**

E 783 [1947, 412.] *Proposed by C. D. Olds, San Jose State College*

Given a parallelogram and its diagonals. Let each side of the parallelogram be divided into $n$ equal parts and let lines be drawn through the points of division, parallel to the sides and to the diagonals of the parallelogram. Find the total number of triangles in the resulting figure.

*Editorial Note.* L. S. Shively pointed out that this essentially is the same as problem 3264 (of this MONTHLY), which he proposed, and whose solution can be found on p. 212 of Vol. 35 (1928). Leo Moser stated that the problem has been treated (at least for special values of $n$) by several puzzlers. Thus, on p. 86 of *Mathematics Clubs and Recreations* by S. I. Jones, is found the case where $n=3$, with the incorrect solution 150. The number of triangles for the general case is

$$\{12n^3 + 18n^2 + 4n - 1 + (-1)^n\}/4.$$

**Salt Solutions**

E 785 [1947, 412]. *Proposed by R. J. Walker, Cornell University*

Each of $n-1$ tanks, $T_1, \cdots, T_{n-1}$, holds $V$ gallons of water, and an $n$th tank, $T_n$, holds $V$ gallons of a salt solution containing $M$ pounds of salt. Liquid is circulated at the rate of $g$ gallons per minute from $T_n$ to $T_{n-1}$, $T_{n-1}$ to $T_{n-2}, \cdots, T_2$ to $T_1$, $T_1$ to $T_n$. How much salt is in $T_n$ after $t$ minutes?

*Solution by Norman Miller, Queen's University, Ontario.* Suppose that tank $T_i$ contains $x_i$ pounds of salt $t$ minutes after the beginning of the operation. In $\Delta t$ minutes thereafter $T_i$ receives from $T_{i+1}$ $gx_{i+1}\Delta t/V$ pounds and loses to $T_{i-1}$ $gx_i\Delta t/V$ pounds, correct to infinitesimals of higher order. This leads to the differential equation $dx_i/dt = g(x_{i+1}-x_i)/V$, or $(D+k)x_i = kx_{i+1}$, where $k = g/V$. The complete set of equations is

$$\begin{aligned}(D+k)x_1 &= kx_2\\ (D+k)x_2 &= kx_3\\ &\cdots\cdots\cdots\\ (D+k)x_n &= kx_1\end{aligned} \tag{1}$$

From (1), by differentiation and substitution, we obtain the following equations, each in one dependent variable:

$$(D+k)^n x_i = k^n x_i, \qquad i = 1, 2, \cdots, n. \tag{2}$$

The set (2) is to be solved subject to the initial conditions: $x_1 = x_2 = \cdots = x_{n-1} = 0$, $x_n = M$, when $t = 0$.

The auxiliary equation of (2) is $(D+K)^n = k^n$, whose roots are $0$, $k(\omega-1)$, $k(\omega^2-1), \cdots, k(\omega^{n-1}-1)$, where $1, \omega, \omega^2, \cdots, \omega^{n-1}$ are the $n$ $n$th roots of unity. It follows that

$$x_1 = \sum_{i=1}^{n} c_i e^{k(\omega^{i-1}-1)t}.$$

The first equation of (1) gives $x_2$ in terms of the same constants:

$$x_2 = \sum_{i=1}^{n} \omega^{i-1} c_i e^{k(\omega^{i-1}-1)t},$$

and similarly for $x_3, \cdots, x_n$. The last equation is

$$x_n = \sum_{i=1}^{n} \omega^{(i-1)(n-1)} c_i e^{k(\omega^{i-1}-1)t}.$$

To find the $c$'s we have when $t = 0$,

$$\sum_{i=1}^{n} c_i = 0$$

$$(3)\qquad \sum_{i=1}^{n} \omega^{i-1} c_i = 0$$

$$\cdots\cdots\cdots\cdots$$

$$\sum_{i=1}^{n} \omega^{(i-1)(n-1)} c_i = M$$

By using the properties $1+\omega+\omega^2+\cdots+\omega^{n-1}=0$, $\omega^n=1$, we get from (3), by addition, $c_1=M/n$. By multiplying the equations of (3) in order by $1, \omega, \omega^2, \cdots, \omega^{n-1}$ and adding, we get $c_n=\omega^{n-1}M/n$. By using multipliers $1, \omega^2, \omega^4, \cdots, \omega^{2(n-1)}$, we get $c_{n-1}=\omega^{n-2}M/n$, etc. Finally, $c_2=\omega M/n$. It follows that

$$x_n = (M/n)\sum_{i=1}^{n} e^{k(\omega^{i-1}-1)t}$$

$$= (M/n)\left[1 + e^{-kt}\sum_{i=1}^{n-1} e^{k\omega^i t}\right].$$

Of the terms in the summation, those equidistant from the ends are either real or conjugate imaginaries. Hence the sum is real and can be expressed without the use of imaginaries. Thus, for $n=2, 3, 4$, the values of $x_n$ reduce to

$$x_2 = (M/2)[1 + e^{-2gt/V}],$$
$$x_3 = (M/3)[1 + 2e^{-3gt/2v}\cos(\sqrt{3}\,gt/2V)],$$
$$x_4 = (M/2)e^{-gt/V}(\cos gt/V + \cosh gt/V).$$

Also solved by Joseph Rosenbaum (partially), Dagnar Dybvik, Hugh Stelson, and the proposer.

The proposer showed that

$$x_n = (M/n)(1 + 2Q) \qquad \text{if } n = 2m + 1,$$
$$x_n = (M/n)(1 + e^{-2gt/V} + 2Q) \qquad \text{if } n = 2m + 2,$$

where

$$Q = \sum_{j=1}^{m} e^{(-1+\cos 2\pi j/n)gt/V}\cos[(\sin 2\pi j/n)gt/V].$$

In either case $x_n$ (and also $x_1, x_2, \cdots, x_{n-1}$) approaches $M/n$ as $t$ increases.

## ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

*Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.*

### PROBLEMS FOR SOLUTION

4290. *Proposed by P. T. Bateman, Yale University*

The function $-\log\left|2\sin\tfrac{1}{2}x\right|$ has the Fourier series

$$\cos x+\tfrac{1}{2}\cos 2x+\tfrac{1}{3}\cos 3x+\cdots.$$

Prove that no partial sum of the series is ever less than $-1$.

4291. *Proposed by J. P. Ballantine, University of Washington, Seattle*

$$\begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \\ 0 & 1 & 2 & \cdots & n-1 \\ 0 & 0 & 1 & \cdots & n-2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{array}$$

Consider the square array with 1's down the principal diagonal, increasing consecutive integers above the diagonal, and zeros below. (a) How many different paths of $i+j-2$ steps are possible from the 1 in the upper left corner to the element in the $i$th row and $j$th column, $j\geqq i$, without passing through any zero element?

(b) Define the value of each path as the product of the elements through which the path passes, not counting the terminal elements. Show that the sum of the values of all the paths in (a) is the coefficient of $(\tan x)^{j-i+1}$ in the $(j+i-2)$nd derivative of $\tan x$ with respect to $x$, expressed in terms of $\tan x$.

4292. *Proposed by R. Goormaghtigh, Bruges, Belgium*

If $s$ and $\rho$ are the arc length and the radius of curvature of a plane curve $\Gamma$ at a variable point $M$, and if it be required that $s$ have a constant ratio to the distance of $M$ from a fixed point, then $\Gamma$ must be a cycloidal curve and

$$\lambda^2 s^2-\rho^2=a^2,$$

$\lambda$ and $a$ being constants.

Prove that, in the case of a twisted curve, the condition is

$$\lambda^2 s^2 - \rho^2 = \left( \int \frac{\rho}{\tau}\, ds + a \right)^2,$$

$\tau$ being the radius of torsion of $\Gamma$ at $M$.

4293. *Proposed by H. F. Sandham, Trinity College, Ireland*

Evaluate

$$\sum_1^\infty \frac{\left(\frac{3-\sqrt{5}}{2}\right)^n}{n^3}.$$

4294. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The lines joining the orthocenter of a triangle to the points of intersection of the medians with the nine-point circle (other than the mid-points of the sides), pass through the vertices of parabolas tangent to two sides of the triangle and having the third side for chord of contact.

## SOLUTIONS

### Feuerbach Hyperbola

4169 [1946, 103]. *Corrected. Proposed by Victor Thébault, Tennie, Sarthe, France*

The tangents at the vertices $A$, $B$, $C$ of a given triangle to its Feuerbach hyperbola form a triangle whose conjugate circle is tangent to the nine-point circle of $ABC$ at its Feuerbach point.

*Solution by the Proposer.**

I. LEMMA: *The polar circle* [1] *of a triangle $A_1B_1C_1$ circumscribed about an equilateral hyperbola $(H)$ and the nine-point circle of the triangle $ABC$, having as its vertices the points of contact of $B_1C_1$, $C_1A_1$, $A_1B_1$, are tangent at the center $\omega$ of $(H)$.*†

Since the equilateral hyperbola $(H)$ is inscribed in triangle $A_1B_1C_1$ and circumscribed about triangle $ABC$, its center $\omega$ is on the polar circle $(h)$ of $A_1B_1C_1$ and on the nine-point circle $(O_9)$ of triangle $ABC$. The two circles $(h)$ and $(O_9)$ are tangent at $\omega$, for if they had another point $\omega'$ in common, there would exist a second equilateral hyperbola $(H')$ passing through $A$, $B$, $C$ and inscribed in $A_1B_1C_1$, which is impossible.

Note. If $(H)$ varies, remaining inscribed in triangle $A_1B_1C_1$, the polar circle $(h)$ of $A_1B_1C_1$ is the envelope of the circle $(O_9)$ of the triangle $ABC$ of the points of contact. If $(H)$ varies, remaining circumscribed about $ABC$, the circle $(O_9)$ of $ABC$ is the envelope of the polar circle $(h)$ of the triangle formed by the tangent lines at $A$, $B$, $C$.

* Translated and checked by W. E. Byrne, Virginia Military Institute.

† V. Thébault, Mathesis, t. LIV, Supplément, p. 38.

II. THEOREM: *The cevians* [2] *of an arbitrary point Q of the equilateral hyperbola (H) circumscribed about a triangle ABC meet the sides BC, CA, AB at D, E, F. If the sides B'C', C'A', A'B' of the orthic triangle A'B'C' of ABC cut the sides EF, FD, DE of triangle DEF at M, N, P, the triangle MNP is circumscribed about the hyperbola (H) touching it at A, B, C, and its polar circle (h) is tangent to the nine-point circle ($O_9$) of triangle ABC at the center ω of (H).*

If the point $Q$ varies on the equilateral hyperbola $(H)$, to a new position $Q_1$ there corresponds a triangle $D_1E_1F_1$, the sides of which pass through the points $M$, $N$, $P$ [3]. The tangents of $(H)$ at $A$, $B$, $C$, which correspond to three particular positions of the point $Q$, pass through $M$, $N$, $P$. Hence the triangle $MNP$ is circumscribed about $(H)$ at $A$, $B$, $C$. From the preceding lemma we conclude that the polar circle $(h)$ of triangle $MNP$ is tangent to the nine-point circle $(O_9)$ of triangle $ABC$ at the center $\omega$ of $(H)$. Thus the theorem is proved.

Note. Triangle $MNP$, the vertices of which are at finite distance, is obtuse-angled.* Hence circle $(h)$ is always real.

III. The Feuerbach hyperbola $(H)$ [4], corresponding to the incenter or to one of the three excenters of triangle $ABC$, contains not only the orthocenter but the corresponding Gergonne point as well. For each Feuerbach hyperbola of triangle $ABC$ the triangle $MNP$ has as its vertices the points of intersection of the sides $B'C'$, $C'A'$, $A'B'$ of the orthic triangle with the sides $EF$, $FD$, $DE$ of the pedal triangle of the incenter (or excenter) with respect to $ABC$. Triangle $MNP$ is circumscribed about $(H)$ at $A$, $B$, $C$, and its polar circle $(h)$ is tangent to the nine-point circle $(O_9)$ of $ABC$ at the center $\omega$ of $(H)$, that is, at the Feuerbach point associated with the incircle (or excircle). It is also known that the incircle (or excircle) is the polar circle $(h)$ of $MNP$.† Our question 4169 leads therefore to a possibly new demonstration of Feuerbach's theorem: *In every triangle the incircle and the excircles are tangent to the nine-point circle.*

We may extend our results by proving the following:

IV. THEOREM: *The polar circle of a triangle $A_1B_1C_1$ circumscribed about an equilateral hyperbola (H), of center ω, is orthogonal to the circle circumscribed about the triangle αβγ, whose vertices are symmetric to ω with respect to $B_1C_1$, $C_1A_1$, $A_1B_1$.*

If we consider a circle $(C)$, the center of which is the orthocenter $H_1$ of triangle $A_1B_1C_1$, the locus of points $P$ in its plane such that the circle passing through the points symmetric to $P$ with respect to $B_1C_1$, $C_1A_1$, $A_1B_1$ is orthogonal to $(C)$ consists of the circumcircle of $A_1B_1C_1$ and of another circle of center $H_1$. The radius of this latter circle $(H_1, \sigma)$, which is the proper locus of $P$, may be determined in the following manner. $(H_1, \sigma)$ cuts $B_1C_1$ at the points $S$, $S'$ of intersection of $B_1C_1$ and the circle of center $A_1$ orthogonal to $(C)$. Let $r$ designate the radius of $(C)$. If $A_1'$ is the foot of the altitude $A_1H_1$ on $B_1C_1$, we have

* Ad. Mineur, Mathesis, 1937, p. 434.

† C. Servais, Mathesis, 1915, p. 134.

$$\overline{SA_1'}^2 = \overline{A_1H_1}^2 - r^2 - \overline{A_1A_1'}^2$$
$$\sigma^2 = \overline{H_1S}^2 = \overline{A_1H_1}^2 - \overline{A_1A_1'}^2 + \overline{H_1A_1}^2 - r^2$$
$$= 2\overline{H_1A_1}\cdot\overline{H_1A_1'} - r^2 = 2\rho^2 - r^2$$

where $\rho$ is the radius of the polar circle of $A_1B_1C_1$. Hence, if we consider a point $P$ of a circle $(C)$ having as its center the orthocenter $H_1$ of a triangle $A_1B_1C_1$ and as its radius $r$, the circle passing through the points $\alpha, \beta, \gamma$ symmetric to $P$ with respect to $B_1C_1$, $C_1A_1$, $A_1B_1$ is orthogonal to the circle $(H_1, \sigma)$. As a special case we conclude that if $P$ is a point of the polar circle $(H_1, \rho)$ of $A_1B_1C_1$, the circle $\alpha\beta\gamma$ is orthogonal to $(H_1, \rho)$. In other words: the circumcircle of triangle $\alpha\beta\gamma$, where $\alpha, \beta, \gamma$ are the symmetric points of the center $\omega$ of the equilateral hyperbola inscribed in triangle $A_1B_1C_1$ with respect to $B_1C_1$, $C_1A_1$, $A_1B_1$, is orthogonal to the polar circle of $A_1B_1C_1$.

By combining Theorems 2 and 4, we obtain the following:

THEOREM: *The tangents at the vertices $A$, $B$, $C$ of a triangle $ABC$ to any of its four Feuerbach hyperbolas form a triangle $A_1B_1C_1$ whose polar circle is tangent to the nine-point circle of $ABC$ at the corresponding Feuerbach point $\phi$ and is orthogonal to the circumcircle of triangle $\alpha\beta\gamma$, the vertices of which are the points symmetric to $\phi$ with respect to $B_1C_1$, $C_1A_1$, $A_1B_1$.*

*Note.* In general, a triangle $A_1B_1C_1$ being given, the locus of the points $P$ in its plane such that the circle passing through the points symmetric to $P$ with respect to $B_1C_1$, $C_1A_1$, $A_1B_1$ is orthogonal to a fixed circle $(C)$ is a bicircular quartic $(K)$ circumscribed about $A_1B_1C_1$. $(K)$ intersects the circumcircle of $A_1B_1C_1$ at the focus $P$ of the parabola of directrix $PH_1$ inscribed in $A_1B_1C_1$. $(K)$ intersects $B_1C_1$ at the points in which the circle of center $A_1$, orthogonal to $(C)$, cuts $B_1C_1$. If $(C)$ is orthogonal to the circumcircle of $A_1B_1C_1$, $(K)$ passes through the orthocenter $H_1$ of $A_1B_1C_1$. If the center of $(C)$ is the orthocenter $H_1$, the quartic degenerates as was stated above.

*Notes by the translator.*

1. Court, College Geometry, p. 155.
2. Court, College Geometry, p. 128.
3. The point $M$ is the pole of $BC$ with respect to $(H)$. If a conic $\Gamma$ is circumscribed about a triangle $ABC$ the necessary and sufficient condition that the pole of $BC$ with respect to $\Gamma$ lie on the side $B'C'$ of the orthic triangle is that the conic $\Gamma$ be an equilateral hyperbola.
4. The equilateral hyperbola determined by the three vertices of a triangle $ABC$ and the incenter (or one of the excenters) is called the Feuerbach hyperbola of $ABC$. There are four of them. The Gergonne point is the point common to the lines drawn from the vertices $A$, $B$, $C$ to the points of contact of the incircle (or of the excircle) with the opposite sides of $ABC$. Usually, in American texts, only the Gergonne point associated with the incircle is known by that name.

**Arc with a Maximum Property**

4232 [1947, 49]. *Proposed by H. D. Ruderman, New York City*

$A: (0, 0)$ and $B: (0, u)$ are joined by the straight line segment $AB$ and a curve

II to enclose a region of area $s$. Let $k$ represent the length of II from $A$ to $B$. Find the equation of II such that the ratio $s/k^2$ is a maximum.

*Solution by Abraham Miller, Washington, D. C.* Let II′ be the reflection of II with respect to $AB$, and consider the resulting closed curve II+II′. Its area, $2s$, cannot exceed the area of a circle of circumference $2k$ (radius $k/\pi$). That is, $2s \leq k^2/\pi$ or $s/k^2 \leq \frac{1}{2}\pi$. The equality sign holds if and only if II+II′ is a circle, and in this case $s/k^2$ is a maximum. Hence II is a semicircle with $AB$ as diameter, the equation being

$$x^2 + y^2 - ay = 0.$$

Also solved by R. C. Buck, R. Lessard, H. J. Zimmerberg, and the Proposer.

**Determinant Divisible by a Power of $x-1$**

4235 [1947, 112]. *Proposed by Irving Kaplansky, University of Chicago, and D. C. Lewis, University of New Hampshire.*

Show that the determinant

$$\begin{vmatrix} (x-1)/1 & (x^2-1)/2 & \cdots & (x^n-1)/n \\ (x^2-1)/2 & (x^3-1)/3 & \cdots & (x^{n+1}-1)/(n+1) \\ \cdots & \cdots & \cdots & \cdots \\ (x^n-1)/n & (x^{n+1}-1)/(n+1) & \cdots & (x^{2n-1}-1)/(2n-1) \end{vmatrix}$$

is a constant times $(x-1)^{n^2}$.

*Solution by Fritz Herzog, Michigan State College.* We shall first prove the following:

THEOREM. *Let $f_1(x), f_2(x), \cdots, f_n(x), g_1(x), g_2(x), \cdots, g_n(x)$ be $2n$ polynomials and let $\phi_{ij}(x) = \int_a^x f_i(u)g_j(u)du$. Then the determinant $|\phi_{ij}(x)|$ vanishes at $x=a$ with a multiplicity of at least $n^2$.*

We have to show that, if $F(x)$ denotes the given determinant, $F^{(k)}(a)=0$ for $0 \leq k < n^2$. Using the well-known method of differentiating a determinant (by rows, say), we obtain

$$F^{(k)}(x) = \sum [k!/(k_1!k_2!\cdots k_n!)] \left| \phi_{ij}^{(k_i)}(x) \right|, \tag{1}$$

where $\sum$ is taken over all sets of non-negative integers $k_1, k_2, \cdots, k_n$ with $k_1+k_2+\cdots+k_n=k$. Let $F^{(k)}(x)=G_k(x)+H_k(x)$, where $G_k(x)$ consists of those terms in (1) in which at least one of the $k_i$ is zero and $H_k(x)$ of those in which all $k_i$ are positive. Since $\phi_{ij}(a)=0$ we have at once $G_k(a)=0$ (which completes the proof when $0 \leq k < n$). We shall now show that $H_k(x)=0$ identically in $x$ and may henceforth omit the variable $x$ in our notation.

Since $\phi_{ij}' = f_i g_j$ we have for $i, j = 1, 2, \cdots, n$ and $k_i \geq 1$

$$\phi_{ij}^{(k_i)} = \sum [(s_i+t_i)!/(s_i!t_i!)] f_i^{(s_i)} g_j^{(t_i)}, \tag{2}$$

where $\sum$ is taken over all non-negative values of $s_i$ and $t_i$ with $s_i+t_i=k_i-1$. Substituting (2) for the elements of $|\phi_{ij}^{(k_i)}|$ in each term of (1) that enters into $H_k$, we obtain a determinant each element of whose $i$th row is written as a sum of $k_i$ terms. Thus we can write this determinant as a sum of $k_1k_2\cdots k_n$ determinants and arrive, after further simplifications, at the following expression for $H_k$:

$$(3)\qquad \sum \frac{kl}{(s_1+t_1+1)\cdots(s_n+t_n+1)}\cdot\frac{f_1^{(s_1)}\cdots f_n^{(s_n)}}{s_1!\cdots s_n!t_1!\cdots t_n!}\cdot|g_j^{(t_i)}|,$$

where $\sum$ is taken over all sets of non-negative integers $s_1, s_2, \cdots, s_n, t_1, t_2, \cdots, t_n$ with $\sum_i(s_i+t_i)=k-n$.

Those terms in (3) in which at least two $t_i$ are equal vanish. In the remaining terms (in which the $t_i$ are distinct) at least two $s_i$ are equal to one another, since otherwise $k-n=\sum_i(s_i+t_i)\geqq n(n-1)$, which contradicts $k<n^2$. These remaining terms of (3) can now be paired according to the following principle: In any such term of (3), let $m$ be the smallest index for which $s_m$ equals one of the other $s_i$ and let $r(>m)$ be the smallest index for which $s_r=s_m$. Then we associate with this term the one with the same values of the $s_i$ and the $t_i$, except that the values of $t_m$ and $t_r$ are interchanged. It is easily seen that this constitutes actually a pairing of the remaining terms of (3) and that the sum of the members of each such pair vanishes. This completes the proof of the Theorem.

We now put $f_i(x)=g_i(x)=x^{i-1}$, $i=1, 2, \cdots, n$, and $a=1$; we obtain $\phi_{ij}(x)=(x^{i+j-1}-1)/(i+j-1)$ and hence, by the Theorem, the determinant given in the problem is divisible by $(x-1)^{n^2}$. It is easily verified that each term in the expansion of this determinant is a polynomial of degree $n^2$ and thus we obtain $|(x^{i+j-1}-1)/(i+j-1)|=A_n(x-1)^{n^2}$, where $A_n=|1/(i+j-1)|$ depends on $n$ only.

Remarks: 1. The Theorem, proved in the above solution, is a generalization of Problem 2 of Part I of the Putnam Mathematical Competition for 1946 (see this MONTHLY, vol. 53 (1946), p. 484).

2. The value of the constant $A_n$ is given by

$$A_1=1;\qquad A_n=\prod_{r=1}^{n-1}(2r+1)^{-1}(C_{2r,r})^{-2},\qquad n=2, 3, \cdots.$$

Also solved by R. P. Brady, H. E. Bray, Robert Breusch, William Gustin, W. J. Harrington, Ivan Niven, J. W. Popow, J. T. Tate, Jr., F. Underwood, John Williamson, and the Proposer.

*Editorial Note.* Most of the other solvers found transformations to establish the equality of the given determinant with the determinant $|(x-1)^{i+j-1}/(i+j-1)|$, whence the proof follows immediately.

**Stern's Diatomic Series**

4236 [1947, 112]. *Proposed by H. D. Grossman, New York City*

Write down two 1's, then a 2 between them, then a 3 between any two numbers whose sum is 3, then a 4 between any two numbers whose sum is 4, and so forth. Prove that the number of $n$'s written down is $\phi(n)$, ($n>1$, $\phi$ is Euler's totient).

*Solution by Leo Moser and R. Steinberg, University of Toronto.* The sequence obtained after the $n$'s have been inserted is precisely the sequence of denominators in the Farey series $F_n$. Since $F_n$ consists of all irreducible fractions from 0 to 1, inclusive, with denominators not exceeding $n$, it will contain $\phi(n)$ irreducible fractions with denominator $n$, so that this will be sufficient to prove the theorem.

If $h/k$, $h''/k''$, $h'/k'$ are three successive terms of $F_n$, and if $h''/k''$ occurs in $F_n$ but not in $F_{n-1}$, then $h+h'=h''$ and $k+k'=k''$. (See Hardy and Wright, *The Theory of Numbers*, second edition, p. 25.) Thus in going from $F_{n-1}$ to $F_n$ we insert terms with denominator $n$ exactly between those terms whose denominators sum to $n$. Hence, by induction, our statement is proved.

Also solved by P. T. Bateman, Howard Eves, William Gustin, L. M. Kelly, Y. S. Luan, and Ivan Niven.

*Editorial Note.* Howard Eves remarks that the sequences here obtained were investigated by Stern in the *Journal für Mathematik*, v. 55, p. 193. The present result is one of many interesting facts proved by Stern. Further study of these sequences was made by D. H. Lehmer, who obtained a curious tie-up with the famous Fibonacci series. Some of Lehmer's results may be found in this MONTHLY [1929, 59–67].

These sequences are also closely related to those treated by Williams and Browne, *A Family of Integers and a Theorem on Circles* (this MONTHLY, 1947, 534–536). Theorem 7 of this article applies, with almost no change, as a proof of the present problem.

## RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

*All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.*

*Compléments de Géométrie Plane.* By R. Deaux. Bruxelles, Maison D'Edition A. de Boeck, 1945. 7+150 pages. 45 Fr.

This small book of 156 pages is devoted to materials comprising that which is generally called College Geometry in America. Its avowed purpose is to help prepare the participants in the competitive examinations for admission to the Polytechnical Faculties of Mons. The three chapters are entitled: I Properties of segments of lines, II Properties of circles, and III Two geometric transformations. Among the topics treated in Chapter I are the usual theorems of Möbius, of Menelaus and Ceva, and those of Papus, Pascal and Brianchon (with respect to the circle). Signed segments, areas of triangles, and angles are used consistently. A rather complete discussion of cross-ratio is given and applications are made to problems of concurrence of lines and collinearity of points and to proofs of Desargues' theorem and its dual. In Chapter II are considered the theorems concerning poles and polars with respect to a circle, radical and antiradical axes, orthogonality of circles and pencils of circles. In Chapter III are treated the similarity transformations and inversion in quite complete detail.

Probably the most important part of the text is that devoted to problems, of which there are 383, grouped according to the sections in the text proper. These vary from quite easy examples of the theory to rather difficult problems.

In general one may say that the text covers considerable, but not all, of the usual topics in College Geometry, but at the same time includes topics often relegated to a chapter on metric applications in our Synthetic Projective Geometries. Definitions of terms are carefully made, and theorems accurately demonstrated.

V. G. GROVE

*Introduction a la Géométrie des Nombres Complexes.* By R. Deaux. Bruxelles, Maison D'Édition A. de Boeck, 1947. 163 pages. 100 Fr.

The chief purpose of this book, as the author explains, is to exploit the use of complex numbers as a tool for geometric investigations. The material in the first part of the book has been used by the author in a course for electrical engineers, and this flavor can be observed in some of the choice of material. However, the writer has chosen to keep the book definitely in the realm of pure geometry rather than to delve into applications, and only a fraction of the material would be of great interest to the engineer.

The first chapter lays the basic groundwork for the discussion, including the usual geometric interpretation of algebraic operations on complex numbers,

and the relation of complex numbers to vectors in the plane. After a brief discussion of the algebraic forms of such simple transformations as reflection, rotation, translation and inversion, the chapter concludes with a consideration of cross ratio. This includes a study of the geometric meaning of harmonic division in the complex plane, and construction problems related to harmonic division and harmonic quadrangles.

In chapter two the techniques developed are applied to problems in plane analytic geometry, and two main topics are selected for consideration. The first is a study of curves that can be generated as a sum of two vectors from a common point, each rotating at constant speed. These are shown to coincide with the cycloidal curves which are described by a fixed point of a moving circular disc as it rolls without slipping on a fixed circle. In particular, the ellipse is shown to be such a curve, and the relation of the geometric properties of the ellipse to the generating vectors developed. The second main topic is that of unicursal curves, that is, curves whose complex equation may be written parametrically as a quotient of polynomials in the real variable $t$. Special emphasis is laid on the cases when the polynomials are linear or quadratic. These curves include the straight line, the circle, all the conics, the unicursal bicircular quartics, and the unicursal circular cubics. Various geometric properties of the curves, such as center and foci of the conics, and double points of the cubics and quartics are neatly obtained in terms of the parametric representations.

Chapter three is devoted to a detailed discussion of the direct circular transformations or homographies and a somewhat less complete consideration of the indirect circular transformations of antihomographies.

It is assumed in the book that the reader has a working knowledge of projective geometry, since the author draws on this freely in places. A more explicit reference to some of the results thus assumed from projective geometry in a few places would add clarity for the casual reader. The book as a whole seems clearly written and typographical errors surprisingly few. While naturally the material considered is not new, the development is interesting, and the book should be a welcome addition to other literature in the field.

S. B. JACKSON

*Elements of Symbolic Logic.* By Hans Reichenbach. The Macmillan Company, 1947. 13+444 pages. $5.00.

This book is intended as a college textbook for a first course in logic. Since the emphasis is more on general ideas, than on the techniques of manipulating symbolic formulas, the work would probably be more suited for a first course in a philosophy, than in a mathematics, department. But it might reasonably be put on a reading list for advanced mathematical students of logic, along with the more recent works of Carnap: mathematicians approach philosophical ideas diffidently, and late in life.

Roughly the first half of the book is taken up with an exposition of the propositional calculus, the lower calculi of predicates and of classes, and the

higher calculus of predicates. This part follows essentially the usual treatment of these topics. A beginning mathematical student (if he knew German) might find it more agreeable to become acquainted with this material through the elegant deductive presentation of Hilbert and Ackermann's *Grundzüge der Theoretischen Logik*. But even here the author sometimes illuminates his topics by flashes of original philosophical insight—so that the mature student will find at least something to learn from these pages.

The remainder of the book is taken up with the analysis of conversational language, and with modalities. In connection with the analysis of conversational language, the author makes the point that the analysis offered by traditional grammar is artificial and inadequate. This inadequacy of traditional grammar rests in turn on the fact that the first grammarians were of a generation trained in the inadequate logic of Aristotle: for there is an intimate connection between logic and grammar. The author attempts to reform traditional grammar in the spirit of modern logic. While much of his discussion here is admittedly exploratory and tentative, he makes some contributions which seem to be of permanent value, particularly in his application to conversational language of Russell's theory of descriptions, and in his treatment of the tenses of verbs (which is an extension of an analysis given by Jespersen). On the other hand, in some cases, the author introduces notation which slavishly imitates terms used in ordinary language, but without giving adequate rules for manipulating such symbols: he sometimes, so to speak, merely translates difficult words of English into symbolic language. An example of this is the notation "$\Theta^*$," which is introduced on page 287 to symbolize the reflexive "this" of English (as when we say, "This sentence consists of six words"). It is clear, that the symbol introduced in this way cannot be governed by the same rules that are assumed for ordinary substantives; for example, from the two true sentences

"This sentence is written in roman type, and contains at least twelve words"

and

"This sentence contains at most seven words"

we cannot deduce the false sentence

"This sentence contains at most seven words, and at least twelve words."

If any light is to be thrown on the use of the reflexive "this" of conversational language, exact rules must be laid down which govern its usage; it is not sufficient merely to replace "this" by another symbol "$\Theta^*$."

The final chapter, which is concerned with modalities, presents an original view of this subject—but a view which is rather close to the views of Carnap, in that, in order to define "necessity," use is made of the notion of provability. A specific criticism of the ideas of this chapter will be found in the last paragraph of the review.

The book is written from the point of view of logical empiricism, a philosophical position which holds that a proposition is meaningful only if it is veri-

fiable as true or false, and that two propositions have the same meaning if they obtain the same verification, as true or false, for all possible observations. The author applies the general point of view of empiricism in a forceful manner: as in the passage on page 356, for example, "we do not wish to say that physical necessity is due to invisible forces tying things together, or that it is a law of reason projected into nature, or whatever else has been subtly devised by certain metaphysicians." Or, to take another example, the brilliant analysis, in terms of grammar, on page 273, of some of the grand philosophical systems of the past: "Disregarding the arbitrariness in the choice of the argument-object has led to an unfortunate absolutism in certain philosophical systems. Thus materialism seems to be guilty of an absolutism of thing-arguments; other philosophical systems are on the search for absolute argument-things in the construction of substances beyond material things. On the other hand, insight into the arbitrariness of the argument has led to the mistake of denying the existence of things. . . . We therefore do not agree with philosophical systems that want to abolish things; instead, we consider the definition of the argument-object a matter of volitional decision."

It is unfortunate that a work so remarkable for philosophical common sense, should suffer from grave technical deficiencies. At many points statements are made, which can hardly stand up under rational examination. In some places the text is almost incomprehensible because of a lack of precision in the introduction of defined terms.

These technical inadequacies are perhaps best illustrated by consideration of the definition, on page 369, of an original nomological statement: "An original nomological statement is an all-statement that is demonstrably true, fully exhaustive, and universal." We notice, in the first place, that the condition that the statement be an all-statement has little force, since, if $A$ is any statement without free variables, then $A$ is equivalent to $(x)A$. Moreover, if we turn back to page 264, to consult the definition of "universal statement," we find it asserted that a statement is not a universal statement if it can be written in a tautologically equivalent form which contains an individual-term. Since for every two statements $A$ and $B$, however, $A$ is tautologically equivalent to $A \cdot (B \supset B)$, we conclude that no statement $A$ is universal (since we can always choose $B$ so that it contains an individual-term)—from which it would follow that original nomological statements themselves do not exist!

J. C. C. McKINSEY

*A Chapter in the Theory of Numbers.* By L. J. Mordell. Cambridge, at the University Press; New York, The Macmillan Company, 1947, 31 pages. $.40.

This little book contains Professor Mordell's inaugural lecture at the University of Cambridge, on the equation $y^2 = x^3 + k$. The solution of this equation, in integers or rationals $x$ and $y$, is an example, perhaps on a par with Fermat's $x^n + y^n = z^n$, of a problem which, at first merely difficult or barren, has ultimately led to interesting and significant developments.

A survey is first made of contributions to the equation by Bachet (1621, $k=-2$ and 17, in rational $x, y$), Fermat (1657, $k=-2$ and 4, in integers), Euler (1738), and others. Fermat's claim to have discovered a "beautiful and subtle method which enabled him to solve such questions in integers," is to be doubted as regards the general equation. Thus, Mordell suggests Fermat could hardly have obtained Nagell's result (1930) that $y^2-17=x^3$ has integral solutions only for $y=3, 4, 5, 9, 23, 282, 375$, and 378661. The first of several examples proved to have no integral solutions, by properties of particular binary quadratic forms and elementary congruences, was that of Lebesgue (1869) who wrote the equation for $k=7$ as $y^2+1=(x+2)(x^2-2x+4)$, and observed that $x+2$ must be positive and of the form $4n+3$. Euler solved $ay^2-kf^2=x^3$ by assuming that $y\sqrt{a}+f\sqrt{k}=(p\sqrt{a}+q\sqrt{k})^3$, whence $y=ap^3+3kpq^2$, $f=3ap^2q+fq^3$; and recognized (without knowing why) that in some cases this assumption fails to give all solutions in integers. The first rigorous proofs of results of some generality were given in 1875, by Pepin, using the Gaussian theory of binary quadratic forms.

A lucid account is given of the bearing on the problem of unique factorization in the quadratic field $R(\sqrt{k})$, and of the connection (suggested above in Euler's example) with the representation of 1 by a binary cubic form. One of the illustrations chosen is $R(\sqrt{6})$, where it is stated there is unique factorization, since the class-number $h=1$. This worried the reviewer for a while, since there are two classes of binary quadratic forms ($x^2-6y^2$ and $2x^2-3y^2$) of this determinant, but serves to illustrate the no longer one–one correspondence between classes of forms and of ideals when $-1$ is not a norm. Then there are twice as many classes of forms as of ideals.

The problem of solving in integers the slightly more general equation $Ey^2=Ax^3+Bx^2+Cx+D$ is reduced to that of solving a finite number of equations of the form $ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4=1$. Applying Thue's Theorem (1908) it follows that $Ey^2=Ax^3+Bx^2+Cx+D$ has (unless the right member has a repeated factor) only a finite number of integral solutions. Mordell's attempt to extend this to $y^2=Ax^4+\cdots+E$ led him to discover his "basis theorem" for the rational points of a general cubic curve, previously conjectured by Poincaré. Weil's generalization to curves of genus $p$ and Siegel's generalization of Thue's Theorem soon followed. The work of Delaunay and Nagell giving bounds to the number of integral solutions, and Fueter's examples of equations with no rational solutions, are mentioned.

This makes an interesting, well-written chapter in the Theory of Numbers, which, in view of the author's concluding remarks, is not yet closed.

GORDON PALL

## NEW BOOKS RECEIVED

*Analytic Geometry*. By R. S. Underwood and F. W. Sparks, Boston, Houghton Mifflin Co., 1947. 10+225 pages. $2.75.

## CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

*Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.*

### CLUB REPORTS, 1946–47

#### Pi Mu Epsilon, University of California at Los Angeles

Papers presented to the California Alpha Chapter of Pi Mu Epsilon, University of California at Los Angeles, included:

*A family of curves from geometrical optics*, by Dr. J. W. Green

*A postulate system for a finite geometry*, by Dr. Leonard Greenstone

*Cryptography*, by Harold Luxenberg

*An empirical equation relating bond order and interatomic distance*, by J. L. Kavanau

*On the solution of some problems in the* MONTHLY, by William Gustin

*Spherical harmonics*, by Milton Drandell

*Dynamical trajectories as geodesics in an n-dimensional Riemannian space*, by Julius Brandstatter

*On certain matrix properties*, by Harold Edmundson.

The annual Pi Mu Epsilon Calculus Prize examination was held this year for the first time since the start of the war. William Seugling was awarded the prize for his examination.

Newly elected officers are: President, Harold Luxenberg; Vice-President, Miss Shirley Marks; Secretary, Raymond Peterson; Treasurer, Dr. W. T. Puckett remains as permanent treasurer; Faculty Adviser, Dr. E. F. Beckenbach.

Retiring officers are: President, Harold Edmundson; Vice-President, Miss Rae Gold; Secretary, Myron Mendelson; Faculty Adviser, Miss Euphemia Worthington.

#### Mathematics Club, Regis College

During the year 1946–47, the Regis College Mathematics Club heard talks by fellow-members who offered topics in widely varied fields which were of particular interest to themselves. The papers were presented as follows:

*Building a doll house*, by Louise Kelley, which was an introduction to the methods and possibilities of descriptive geometry

*The problem of Apollonius*, by Clare Glennon

*Models in plastic*, by Virginia Harris

*Slope in solid analytic geometry*, by Virginia Demeo—discussion of an article in this MONTHLY, May, 1946

*Sir William Hamilton's icosian game*, by Catherine Walsh—discussion of an article in this MONTHLY

*Mascheroni's constructions*, or *the geometry of the compasses*, by Rita Dailey.

This lecture was accompanied by delineascope projections showing the constructions by an ingenious use of colored circles.

Members also attended bi-weekly meetings of the Harvard College Mathematics Club where talks were given by graduate students and professors. Senior members attended the meetings of the New England Mathematics Teachers' Association. The program of the Spring Meeting of the Association included a lecture and exhibit of the giant calculating machine at Harvard College.

A sizable addition was contributed by the club to the Regis collection of mathematical models. The students worked in strings and plastic to produce several interesting surfaces and combinations of surfaces.

To celebrate the tenth anniversary of the foundation of the club, alumnae members established a fund whereby each year the student with the highest average in mathematics over a four-year period would be awarded a prize of twenty-five dollars at graduation. This announcement was made at a tea planned in recognition of the occasion.

The social calendar of the year included a Christmas party, a supper party in May in honor of the Senior members, and the tenth anniversary tea under the auspices of the mathematics alumnae.

Officers for the year were the following: President, Rita Dailey; Vice-President, Virginia Harris; Secretary, Virginia Demeo; Treasurer, Bernadette Vitti.

**Mathematics Club, University of Colorado**

The Mathematics Club at the University of Colorado was revived October 28, 1946 after a lapse of several years because of the war. The following talks were presented:

*Hyperbolic functions*, by Mr. W. R. Hiatt

*Stirling's approximation-formula*, by Mr. W. W. Mitchell, Jr.

*Derivation of some functions from Gauss' hypergeometric series*, by Mr. G. A. Culpepper

*Finite differences*, by Mr. F. R. Poole and Professor A. B. Farnell

*Some properties of the cycloid*, by Mr. B. Hunt

*Linear diophantine equations*, by Mr. H. G. H. Bartram

*Wallis's product formula*, by Mr. W. W. Mitchell, Jr. This paper was particularly interesting because Mr. Mitchell translated from Wallis's original work published in Latin, and also showed how Wallis computed the areas under various curves.

*The transcendence of e*, by Mr. H. G. H. Bartram

*The Laplace transform*, by Mr. DeVol

*Euler's $\phi$-function*, by Mr. B. Hunt

Officers for 1946–47 were: President, Mr. F. R. Poole; Secretary-Treasurer, Mr. H. G. H. Bartram.

Officers elected for 1947–48 are: President, Mr. W. W. Mitchell, Jr.; Secretary-Treasurer, Mr. B. Hunt.

### Newtonian Society, Lehigh University

The Newtonian Society, freshman honorary mathematics society at Lehigh University, held five meetings during the school year 1946–47.

Speakers and topics were:

*Curve fitting*, by Professor V. V. Latshaw

*Euler's formula and the regular polyhedra*, by Professor R. R. Stoll

*Mapping*, by Professor J. O. Chellevold

*What is a number?*, by Professor T. Hailperin

Officers for the year were: President, D. N. Love; Vice-President-Treasurer, J. M. Christie; Secretary, H. Dowling.

---

## NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.*

### A.A.A.S. CENTENNIAL CELEBRATION

Dr. Harlow Shapley, Chairman of the Centennial Policy Committee, has announced that the Centennial Celebration of the American Association for the Advancement of Science will be held on September 13–17, 1948 in Washington, D. C. The keynote of the Centenary will be "One World of Science."

### SEVENTH INTERNATIONAL CONGRESS OF APPLIED MECHANICS

The Seventh International Congress of Applied Mechanics will be held at the Imperial College of Science and Technology, South Kensington, London, England, on September 5–11, 1948.

Applications and other material concerning the Congress may be secured by writing to the Secretary of the American Mathematical Society, University of Pennsylvania, Philadelphia 4, Pennsylvania.

### BACK COPIES OF THE "MONTHLY"

During the past two years, the membership of the Association has grown at such a rapid rate that it has been difficult to estimate in advance the proper number of copies of the MONTHLY which were to be printed each month.

As a result, the office of the Association finds that it is lacking sufficient copies of the MONTHLY for 1946 (especially January, May, June–July and August–September), 1947 (especially January and December) and January 1948.

The Association will be glad to refund postage to any member who wishes to dispose of the above-mentioned numbers of the MONTHLY by contributing them

to the Association. Mail to: Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

## PERSONAL ITEMS

The Duodecimal Society of America has conferred the 1948 Annual Award upon Mr. H. C. Robert of Atlanta, Georgia.

The Institute of Mathematical Statistics has announced the election of the following officers: Professor Abraham Wald, head of the Department of Mathematical Statistics of Columbia University, as president; Dr. Churchill Eisenhart, chief of the Statistical Engineering Laboratory of the National Bureau of Standards, as a vice-president; Professor Henry Scheffé of the University of California at Los Angeles, as a vice-president.

The Royal Society of London has awarded the Copley Medal to the late Professor Emeritus G. H. Hardy of the University of Cambridge for his part in the development of mathematical analysis in Britain during the last thirty years.

Associate Director Joseph Slepian of Westinghouse Research Laboratories has received the Edison medal for 1947 from the American Institute of Electrical Engineers.

Professor Hassler Whitney of Harvard University received from Yale University in 1947 an honorary doctorate of science.

Professor R. L. Wilder of the University of Michigan has been elected vice-president of Section A of the American Association for the Advancement of Science.

Associate Professor J. W. T. Youngs of Indiana University has received a Guggenheim fellowship.

Associate Professor E. W. Anderson of Iowa State College of Agriculture and Mechanic Arts has been promoted to a research professorship in mechanical engineering and a professorship in mathematics.

Assistant Professor H. G. Apostle of Amherst College has been appointed to an assistant professorship in the Department of Philosophy of the University of Chicago.

Mr. L. C. Bagby of the Jam Handy Corporation has been appointed to a professorship at the Lawrence Institute of Technology, Detroit, Michigan.

Mr. J. W. Beach, Iowa State College of Agriculture and Mechanic Arts, has been promoted to an assistant professorship.

Professor Arne Beurling will be a visiting lecturer at Harvard University for the coming academic year.

Mr. F. E. Bortle, Iowa State College of Agriculture and Mechanic Arts, has been promoted to an assistant professorship.

Associate Professor A. T. Brauer of the University of North Carolina has been promoted to a professorship.

Assistant Professor A. B. Brown of Queens College has been promoted to an associate professorship.

Professor C. L. Buxton is serving as Director of the Malone Branch of Clarkson College of Technology.

Professor Henri Cartan of the University of Paris is a visiting lecturer at Harvard University this term.

Mr. P. T. Copp is now a research engineer with the Hays Corporation, Michigan City, Indiana.

Dr. C. L. Dolph, University of Michigan, has been promoted to an assistant professorship.

Professor Philip Franklin of the Massachusetts Institute of Technology is also serving as visiting lecturer on mathematical physics at Harvard University during the current spring term.

Assistant Professor G. N. Garrison, Lehigh University, has been promoted to an associate professorship.

Professor V. G. Grove of Michigan State College will be visiting professor at the University of Puerto Rico during the present term.

Associate Professor E. R. Heineman of Texas Technological College has been promoted to a professorship.

Assistant Professor Rufus Isaacs of the University of Notre Dame has accepted a position as research engineer with the North American Aviation Corporation, Los Angeles, California.

Mr. P. B. Johnson of the University of Illinois has been appointed to an assistant professorship at Occidental College.

Mr. C. H. Lindahl, Iowa State College of Agriculture and Mechanic Arts, has been promoted to an assistant professorship.

Professor C. E. Love of the University of Michigan has retired.

Mr. C. R. Morris of Indiana University has accepted a position as mathematician with the National Union Radio Corporation, Orange, New Jersey.

Mr. Bob Parker, Texas Technological College, has been promoted to an assistant professorship.

Assistant Professor Sallie Pence of the University of Kentucky has been promoted to an associate professorship.

Associate Professor D. E. South, University of Kentucky, has been promoted to a professorship.

Assistant Professor R. R. Stoll of Lehigh University has been promoted to an associate professorship.

Professor Alessandro Terracini of the University of Tucumán has returned to the University of Turin.

Professor Gerhard Tintner of Iowa State College of Agriculture and Mechanic Arts will be on leave of absence during the coming academic year and will attend the Institute of Economics at Cambridge University.

Dr. C. A. Truesdell, Research Department, Naval Ordnance Laboratory, White Oak, Maryland, has been promoted to the position of chief of the theoretical mechanics section.

Assistant Professor F. E. Ulrich, Rice Institute, has been promoted to an associate professorship.

Professor J. L. Walsh of Harvard University will be on leave of absence during this term.

Professor D. V. Widder of Harvard University will be on leave of absence during the coming academic year.

The following appointments to instructorships are announced:

New York State College for Teachers, Buffalo: Mr. Rudolph Cherkauer.

Texas Technological College: Mrs. Susie Kammerdiener, Miss Lillian McGlothlin, Mr. E. H. Thomas, Mrs. Naomi Thompson.

University of Illinois, Chicago Undergraduate Division: Mr. Furio Alberti.

University of Kentucky: Miss Elsie T. Church, Mrs. L. C. Cooper, Mr. D. L. Daly, Mr. D. C. Rose, Miss Genevieve Snider.

University of Pennsylvania: Miss Jean B. Walton.

Dr. Mellen W. Haskell, formerly head of the Department of Mathematics of the University of California, died January 15, 1948 at the age of eighty-four years.

Professor Emeritus Vladimir Karapetoff of Cornell University died January 11, 1948.

---

## THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### REPORT OF THE TREASURER FOR THE YEAR 1947

The following report of Professor W. B. Carver as Treasurer for the year 1947 has been approved by the Finance Committee and accepted by vote of the Board of Governors.

### I. Total Funds of the Association on December 31, 1946

(See Treasurer's report, pp. 249–252 of the Monthly for April, 1947)

| | | |
|---|---|---|
| Current Fund (checking account) | | $ 6,348.92 |
| Savings Account (Ithaca Savings Bank) | | 1,022.66 |
| Invested Funds (Cleveland Trust Company) | | |
| Carus Fund | $10,513.65 | |
| Chace Fund | 10,000.06 | |
| Houck Fund | 9,382.63 | |
| Chauvenet Fund | 665.58 | |
| Life Membership Fund | 648.60 | |
| General Fund | 25,659.78 | 56,870.30 |
| | | $64,241.88 |

## II. Current Fund Account

| Receipts | | Expenditures | |
|---|---|---|---|
| Balance, Jan. 1, 1947 | $ 6,348.92 | Monthly | |
| Dues | 11,617.05 | Publication | $ 7,828.50 |
| Initiation fees | 656.00 | Reprints | 223.92 |
| Subscriptions to the Monthly | 3,685.14 | Editor-in-Chief's Office | 528.39 |
| Sale of back numbers Monthly | 915.50 | Secretary-Treasurer's Office | |
| Advertisements | 1,109.50 | Clerical help | 3,650.16 |
| Interest on General Fund | 919.02 | Postage | 402.81 |
| Interest on Carus Fund | 376.07 | Printing | 434.53 |
| Interest on Chace Fund | 358.17 | Office supplies | 282.97 |
| Interest on Houck Fund | 336.67 | Bank fee | 100.00 |
| Interest on Chauvenet Fund | 23.64 | Moving office | 210.82 |
| Interest on Life Membership Fund | 23.28 | Exec. and Finance Committees | 799.27 |
| Interest from Hardy Fund | 120.00 | Coordinating Committee | 19.66 |
| Sale of Archibald's Outline | 210.13 | Coop. Committee on Teaching | 108.30 |
| Sales of Monographs (Carus) | 2,146.43 | Sections and Regions | 387.75 |
| Sale of Papyrus (Chace) | 230.00 | Subventions | |
| Sale of Slaught Memorial Paper I (Chace) | 14.00 | American Math. Society | 100.00 |
| Sales of exchange periodicals | 28.60 | Mathematical Reviews | 350.00 |
| Miscellaneous sources | 3.56 | Amer. Council on Educ. | 180.76 |
| Transferred from Chace Fund | 494.54 | Back numbers Monthly | 225.12 |
| Transferred from General Fund | 901.92 | Bank exchange charges | 6.41 |
| | | Reprinting Monos. 6 & 7 (Carus) | 1,509.63 |
| | | Printing Slaught Mem. Paper I (Chace) | 1,096.71 |
| | | Handling exchanges of periodicals | 76.00 |
| | | Com. on Aid to Devastated Libraries | 250.00 |
| | | Mrs. B. F. Finkel (Hardy Fund) | 120.00 |
| | | Half cost of "Smith list" | 120.49 |
| | | Addition to Trust Fund | 6,500.00 |
| | | Transferred to Carus Fund | 1,012.87 |
| | | Transferred to Houck Fund | 336.67 |
| | | Transferred to Chauvenet Fund | 23.64 |
| | | Transferred to Life Membership Fund | 23.28 |
| | | Balance, Dec. 31, 1947 | 3,609.48 |
| | $30,518.14 | | $30,518.14 |

## III. Savings Account, Ithaca Savings Bank

| | | | |
|---|---|---|---|
| Balance, Jan. 1, 1947 | $ 1,022.66 | | |
| Interest | 23.16 | Balance, Dec. 31, 1947 | $ 1,045.82 |
| | $ 1,045.82 | | $ 1,045.82 |

## IV. Invested Funds, Cleveland Trust Company

| | | | |
|---|---|---|---|
| Cash Balance, Jan. 1, 1947..... | $ 174.30 | Decrease in values of securities.. | $ 2,536.00 |
| Values of Securities, Dec. 31, 1946 | 56,696.00 | Values of securities, Dec. 31, 1947 | 60,635.00 |
| From Current Fund..... | 6,500.00 | Cash bal., Dec. 31, 1947..... | 199.30 |
| | $63,370.30 | | $63,370.30 |

### List of Securities

| | Par Value | Market Value Dec. 31, 1947 |
|---|---|---|
| U.S. Treasury Bonds, 2%, 1950..... | $3,000.00 | $ 3,000.00 |
| U.S. Treasury Bonds, 2%, 1954..... | 2,000.00 | 2,020.00 |
| U.S. Treasury Bonds, 2½%, 1969..... | 2,000.00 | 2,020.00 |
| U.S. Treasury Bonds, 2½%, 1972..... | 1,000.00 | 1,000.00 |
| U.S. Treasury Bonds, 2¼, 1962..... | 5,000.00 | 5,000.00 |
| U.S. Treasury Bonds, 1¾%, 1948..... | 2,000.00 | 2,000.00 |
| U.S. Savings Bonds, Ser. G, 2½%, 1953..... | 3,000.00 | 2,850.00 |
| U.S. Savings Bonds, Ser. G, 2½%, 1954..... | 8,200.00 | 7,790.00 |
| U.S. Savings Bonds, Ser. G, 2½%, 1958..... | 3,000.00 | 2,910.00 |
| U.S. Savings Bonds, Ser. G, 2½%, 1959..... | 6,500.00 | 6,435.00 |
| Canadian Nat. Ry. Co. Bonds, 4½%, 1956..... | 2,000.00 | 2,220.00 |
| Amer. Tel & Tel Co. Conv. Deb. Bonds, 2½%, 1957..... | 500.00 | 540.00 |
| C. and O. Ry. Co. Ref. Bonds, Ser. D, 3½%, 1996..... | 3,000.00 | 3,030.00 |
| Columbus and So. Ohio Elec. Co. Bonds, 3¼%, 1970..... | 2,000.00 | 2,100.00 |
| New York Steam Corp. 1st Mort. Bonds, 3⅓%, 1963..... | 1,000.00 | 1,040.00 |
| Amer. Tobacco Co. Bonds, 3%, 1969..... | 4,000.00 | 4,000.00 |
| C. & O. Ry. Co. common stock, 25 sh..... | | 1,075.00 |
| Amer. Tel. & Tel. Co. common stock 30 sh..... | | 4,530.00 |
| Standard Oil Co. of New Jersey com. stock, 20 sh..... | | 1,580.00 |
| Atch. Top. and Santa Fe. R. R. pfd stock, 15 sh..... | | 1,515.00 |
| Commonwealth Edison Co. common stock, 80 sh..... | | 2,160.00 |
| Dana Corp. cum. pfd. stock, 20 sh..... | | 1,820.00 |
| | | $60,635.00 |

## V. Carus Fund

| | | | |
|---|---|---|---|
| Balance, Jan. 1, 1947..... | $10,513.65 | Reprinting Monographs..... | $ 1,509.63 |
| Sale of Monographs..... | 2,146.43 | Decrease in value of securities... | 468.22 |
| Interest..... | 376.07 | Balance, Dec. 31, 1947..... | 11,058.30 |
| | $13,036.15 | | $13,036.15 |

## VI. Chace Fund

| | | | |
|---|---|---|---|
| Balance Jan. 1, 1947..... | $10,000.06 | Printing Slaught Mem. Paper I.. | $ 1,096.71 |
| Sale of Papyrus..... | 230.00 | Decrease in values of securities.. | 445.93 |
| Sales of Slaught Mem. Paper I.. | 14.00 | Balance, Dec. 31, 1947..... | 9.059,59 |
| Interest..... | 358.17 | | $10,602.23 |
| | $10,602.23 | | |

## VII. Houck Fund

| | | | |
|---|---|---|---|
| Balance, Jan. 1, 1947 | $ 9,382.63 | Decrease in value of securities | $ 419.18 |
| Interest | 336.67 | Balance, Dec. 31, 1947 | 9,300.12 |
| | $9,719.30 | | $9,719.30 |

## VIII. Chauvenet Fund

| | | | |
|---|---|---|---|
| Balance, Jan. 1, 1947 | $ 665.58 | Decrease in value of securities | $ 29.43 |
| Interest | 23.64 | Balance, Dec. 31, 1947 | 659.79 |
| | $ 689.22 | | $ 689.22 |

## IX. Life Membership Fund

| | | | |
|---|---|---|---|
| Balance, Jan 1, 1947 | $ 648.60 | To General Fund | $ 44.30 |
| Interest | 23.28 | Decrease in value of securities | 28.98 |
| | | Balance, Dec. 31, 1947 | $ 598.60 |
| | $ 671.88 | | $ 671.88 |

## X. General Fund

| | | | |
|---|---|---|---|
| Balance, Jan. 1, 1947 | $25,659.78 | Decrease in value of securities | $ 1,144.26 |
| From Current Fund | 6,500.00 | Transferred to Current Fund | 901.92 |
| From Life Membership Fund | 44.30 | Balance, Dec. 31, 1947 | $30,157.90 |
| | $32,204.08 | | $32,204.08 |

## XI. Total Funds of the Association, December 31, 1947

| | | |
|---|---|---|
| Current Fund (checking account) | | $ 3,609.48 |
| Savings Account (Ithaca Savings Bank) | | 1,045.82 |
| Invested Funds (Cleveland Trust Company) | | |
| Carus Fund | $11,058.30 | |
| Chase Fund | 9,059.59 | |
| Houck Fund | 9,300.12 | |
| Chauvenet Fund | 659.79 | |
| Life Membership Fund | 598.60 | |
| General Fund | 30,157.90 | 60,834.30 |
| | | $65,489.60 |

## DECEMBER MEETING OF THE TEXAS SECTION

A special meeting of the Texas Section of the Mathematical Association of America was held at Texas Christian University, Fort Worth, Texas, on Friday and Saturday, December 6–7, 1946. Vice-Chairman R. S. Underwood presided.

The following members of the Association were present: H. E. Bray, H. F. Bright, Myrtle Brown, J. E. Burnam, J. V. Cooke, Alice Dean, Elizabeth Davis, H. J. Ettlinger, R. E. Greenwood, E. R. Heineman, J. E. Howard, Harlan Miller, J. W. Querry, W. A. Rees, D. W. Starr, E. J. Stulken, F. E. Ulrich, R. S. Underwood, W. M. Whyburn, Mabel Williams, H. E. Woodward.

At the business meting it was decided that the present officers should continue until the annual meeting in the spring. The annual meeting will be held at Texas Technological College. The officers are: Chairman, E. H. Hanson, North Texas State Teachers College; Vice-Chairman, R. S. Underwood, Texas Technological College; Acting-Secretary, C. R. Sherer, Texas Christian University.

An informal dinner was held at the Worth Hotel Friday evening. At this time the Section heard an address entitled *The Role of Mathematics in the Post World War II Educational Programs*, by W. M. Whyburn, President, Texas Technological College.

The following papers were given at the meetings of the Section:

1. *A plane version of solid analytical geometry*, by R. S. Underwood, Texas Technological College.

This paper adds to the results obtained in the article entitled *An Analytic Geometry for n Variables*, this MONTHLY, vol. 52, 1945, pp. 253–262. In particular it deals with the alternative to solid analytic geometry provided by the three-axes plane, suggesting briefly the relative advantages of the new system for certain special purposes. Subjects discussed included generalized coördinates, the straight line, point-to-point and point-to-line formulas, and applications to algebra and calculus.

2. *Note on the zeros of* $P_n^m(\cos\theta)=0$ and $dP_n^m(\cos\theta)/d\theta=0$, *considered as functions of* $n$, by C. W. Horton, Defense Research Laboratory Staff, University of Texas, introduced by H. J. Ettlinger.

In many physical problems in which the boundary conditions are specified over the surface of a cone, it is necessary to know the roots of the equations

$$P_n^m(\cos\theta) = 0 \tag{1}$$

and

$$\frac{d}{d\theta} P_n^m(\cos\theta) = 0 \tag{2}$$

considered as functions of $n$. This problem has been solved by Bholanath Pal. He develops infinite series for the roots which converge rapidly and are very suitable for numerical computation. In deriving his solution Pal introduced a parameter $k$ which takes on successive integral values and thereby yields successive roots of the equations.

It is the purpose of this note to point out that the value $k=1$ with which Pal commenced the series does not always give the first root of the equation, and sometimes it gives a number which is not a root of the equation. A corrected table of roots based on the calculations of Pal is given.

3. *Recent high speed methods of computation developed for war research*, by H. J. Ettlinger, University of Texas.

The tremendous program of technical development of the past six or seven years has resulted in an accelerated growth in the use of mechanical and electronic devices for carrying out extensive and complicated computations arising in scientific fields. The nature of the problems treated extends from that of solving linear algebraic equations in a substantial number of unknowns to obtaining solutions of systems of ordinary differential equations, partial differential equations, integral equations, and integro-differential equations. The results obtained are not limited to a narrow practical field of obtaining numerical answers to a desired number of decimal places, but it is envisaged that these devices will be helpful in theoretical problems, particularly in relation to non-linear systems.

4. *Riemann surfaces, aspects of the type problem*, by F. E. Ulrich, Rice Institute.

5. *On the roots of the derivatives of a complex polynomial*, by H. E. Bray, Rice Institute.

Let

$$P(z) = z(z - z_1)(z - z_2) \cdots (z - z_n)$$
$$Q(x) = x(x - |z_1|)(x - |z_2|) \cdots (x - |z_k|)(x + |z_{k+1}|)$$
$$P'(z) = \frac{dP}{dz}, \qquad Q'(x) = \frac{dQ}{dx}$$

where $z_1 < z_2 < \cdots < z_n$. The real polynomial $Q'(x)$ has $k$ positive roots $\xi_1 < \xi_2 < \cdots < \xi_k$, and $n-k$ negative roots, $\xi_n < \xi_{n-1} < \cdots < \xi_{k+1}$. It is proved that, if $|\xi_{k+1}| > \xi_k$, then $P'(z)$ has exactly $k$ roots in the circle $|z| \leq \xi_k$ and $n-k$ roots in the region $|z| \geq \xi_{k+1}$.

6. *Long range navigation*, by R. E. Greenwood, University of Texas.

7. *Some pages from the history of trigonometric analysis*, by R. E. Langer, University of Texas.

This was an extended address delivered at the invitation of the Section.

8. *Round table discussion of teaching problems in mathematics*, by Elizabeth Dice, Mable Williams, and W. L. Porter of Texas A. and M., all introduced by the Secretary.

C. R. SHERER, *Acting Secretary*

## APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at the Texas Technological College, Lubbock, Texas, on Friday and Saturday, April 25–26, 1947. Professor E. H. Hanson, Chairman of the Section, presided.

The following members of the Association were present: Ina Mae Bramblett, H. E. Bray, Myrtle Brown, H. D. Brunk, J. E. Burnam, J. V. Cooke, H. J. Ettlinger, E. H. Hanson, E. R. Heineman, P. E. Lerret, J. N. Michie, Elva Miller, Harlan Miller, C. R. Sherer, F. W. Sparks, F. E. Ulrich, R. S. Underwood, H. S. Wall, G. T. Whyburn.

At the business meeting the following officers were elected for next year: Chairman, R. S. Underwood, Texas Technological College; Vice-Chairman, F. E. Ulrich, Rice Institute; Secretary-Treasurer, C. R. Sherer, Texas Christian University. The location of the next meeting was not determined.

An informal dinner was held at the Hilton Hotel on Friday, April 25, after which an address was given by Professor H. J. Ettlinger.

The following papers were presented on Saturday.

1. *Regions in which an analytic function is flat*, by F. E. Ulrich, Rice Institute.
2. *The strong law of large numbers*, by H. D. Brunk, Rice Institute.
3. *Extended analytic geometry*, by R. S. Underwood, Texas Technological College.
4. *A general criterion for convergence of Fourier series*, by H. E. Bray, Rice Institute.
5. *Application of continued fractions to the location of roots of polynomials*, by H. S. Wall, University of Texas.

This was an invited address of an hour's duration.

6. *Round table discussion on examination papers*, by E. R. Heineman, Texas Technological College, and C. R. Sherer, Texas Christian University.

C. R. SHERER, *Secretary*

## CALENDAR OF FUTURE MEETINGS

Thirtieth Summer Meeting, Madison, Wisconsin, September 6–7, 1948.

Thirty-second Annual Meeting, Columbus, Ohio, December 31, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pennsylvania State College, Pennsylvania, May 8, 1948

ILLINOIS, Illinois Institute of Technology, Chicago, May 14–15, 1948

INDIANA, Purdue University, West Lafayette, May 8, 1948

IOWA, Fairfield, April 16–17, 1948

KANSAS, Atchison, April 10, 1948

KENTUCKY, Berea, May, 1948

LOUISIANA-MISSISSIPPI, Southwestern Louisiana Institute, Lafayette, Louisiana, April 23–24, 1948

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, United States Naval Academy, Annapolis, Maryland, May 8, 1948

METROPOLITAN NEW YORK, Washington Irving High School, April 24, 1948

MICHIGAN, University of Michigan, Ann Arbor, April 3, 1948

MINNESOTA, College of St. Thomas, St. Paul, May 8, 1948

MISSOURI, University of Kansas City, Kansas City, April 23, 1948

NEBRASKA, University of Nebraska, Lincoln, May 1, 1948

NORTHERN CALIFORNIA, San Francisco, January 29, 1949

OHIO, Ohio State University, Columbus, April 3, 1948

OKLAHOMA

PACIFIC NORTHWEST

PHILADELPHIA, Philadelphia, Pa., November 27, 1948

ROCKY MOUNTAIN, April 23–24, 1948

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN, New Mexico Highlands University, Las Vegas, New Mexico, May 3–6, 1948

TEXAS, Rice Institute, Houston, April 23–24, 1948

UPPER NEW YORK STATE, Union College, Schenectady, May 1, 1948

WISCONSIN, Beloit, May 8, 1948

# *new H. M. Co. texts*

## MATHEMATICS FOR USE IN BUSINESS

**by *C. E. Hilborn*, Assistant Professor of Business Administration, School of Business Administration, Duquesne University.**

*Mathematics for Use in Business* provides material for a thorough-going first course in business mathematics, well suited to freshman courses in schools of business administration or terminal courses where "practical" mathematics is indicated. This text is the outgrowth of twelve years' teaching of business mathematics in Duquesne University. It has been used in various experimental editions with freshmen in business administration and with evening school students. The present edition shows the finished presentation which comes from extensive use and constant rewriting for improvement. Throughout the book the author treats his subjects with thoroughness to meet the most rigorous requirements of any first course and with a style which will gain and hold the interest of the class.

## ANALYTIC GEOMETRY

**by *R. S. Underwood* and *Fred W. Sparks*, Professors of Mathematics, Texas Technological College.**

In *Analytic Geometry* the authors have produced a brief text possessing clarity, serviceability, and efficiency. The book includes only the most immediately useful topics. New concepts are introduced as they are needed in the normal development of the subject, with new proofs for traditionally difficult subjects. Though the departure from classical procedures are numerous, at no point have the authors adopted a novel approach merely for the sake of the change. A large number of carefully selected and graded problems are included.

## BASIC MATHEMATICS: A WORKBOOK

**by *M. Wiles Keller*, Associate Professor of Mathematics, Purdue University, and *James H. Zant*, Professor of Mathematics, Oklahoma Agricultural and Mechanical College.**

The successful approach of this workbook is based upon (1) the discovery through a testing program of the topics which need attention; (2) the provision of a minimum yet adequate amount of explanation; (3) the use of step-by-step illustrations to accompany rules; (4) the provision of a large number of problems.

**HOUGHTON MIFFLIN COMPANY**

Boston New York Chicago Dallas San Francisco

*An entirely new Hart text*

## Intermediate Algebra for Colleges

This new intermediate algebra gives students a collegiate substitute for third semester high school algebra—

It is specifically designed as a text for a course to precede classical college algebra, or for a terminal course for students who will need merely substantial knowledge of the elements of algebraic technique and computation in college—

Appropriate refresher work in arithmetic and approximate computation is given in an early chapter. Skill in computation is emphasized continuously throughout the book—

Written in a style suitable to the maturity of college students but with the assumption that all the fundamentals of the subject must be introduced to them anew—

276 pages of text — Price $2.50

## By WILLIAM L. HART

*D. C. Heath and Company*

Boston New York Chicago Atlanta San Francisco Dallas London

# WILEY

## *Students "go for" these textbooks!*

### UNIFIED CALCULUS

By Edward S. Smith, *Professor of Mathematics,* Meyer Salkover, *Associate Professor of Mathematics, and* Howard K. Justice, *Professor of Mathematics and Assistant Dean; all at the University of Cincinnati.*

An ideal mathematics textbook which puts particular emphasis on the application of calculus to both physics and mechanics. Differential and integral calculus are successfully blended throughout the textbook. *1947. 510 pages. $3.50.*

### CALCULUS

By Edward S. Smith, Meyer Salkover, and Howard K. Justice

For use in both liberal arts and engineering colleges, this book is designed for a first course in calculus, presupposing a knowledge of freshman mathematics. *1938. 558 pages. $3.50.*

### CALCULUS

By Frederick H. Miller, *Professor and Head of Department of Mathematics, The Cooper Union School of Engineering.*

A textbook for students of pure mathematics or those who will use calculus in scientific or engineering courses. The treatment has been clarified in the second edition. *1946. 416 pages. $3.50.*

### COLLEGE ALGEBRA AND TRIGONOMETRY

By Frederick H. Miller

This is a new book—an integrated treatment of the two subjects—which develops for the student a suitable foundation for the study of analytic geometry and calculus. *1945. 324 pages. $3.25.*

### ANALYTIC GEOMETRY

By Edward S. Smith, Meyer Salkover, and Howard K. Justice

Covers topics commonly taught in analytical geometry and offers training in the phases of the subject that will fit the student for further studies in mathematics, science, and engineering. *1943. 298 pages. $2.75.*

### INTERMEDIATE DIFFERENTIAL EQUATIONS

By Earl D. Rainville, *Assistant Professor of Mathematics, University of Michigan.*

Designed to make the transition less abrupt between elementary and advanced courses, this book furnishes the student with further tools for constructing computable solutions for specific differential equations. *1943. 213 pages. $3.00.*

### A TREATISE ON ADVANCED CALCULUS

By Philip Franklin, *Professor of Mathematics, Massachusetts Institute of Technology.*

Including those portions of the theory of functions of real and complex variables necessary for a logical presentation of the methods of the infinitesimal calculus and their applications to geometry and physics. *1940. 593 pages. $6.50.*

### A FIRST YEAR OF COLLEGE MATHEMATICS

By Henry J. Miles, *Associate Professor of Mathematics, University of Illinois.*

A well-rounded textbook for a two-semester, freshman course in college mathematics. It includes college algebra, plane trigonometry, plane and solid analytical geometry, and also some topics of differential calculus. *1941. 607 pages. $3.25.*

### VECTOR ANALYSIS

By Henry B. Phillips, *Head of Mathematics Department, Massachusetts Institute of Technology.*

This book consists of two parts, the first covering the elementary principles and fundamental operations and the second, an analysis of fields, the properties of potentials, and linear functions. *1933. 236 pages. $2.75.*

---

**JOHN WILEY & SONS, Inc.**

**440 Fourth Avenue**

**New York 16, New York**

# 4 Important McGraw-Hill Books

## SOLID GEOMETRY

*By James Sutherland Frame, Michigan State College. Ready in spring*

• Departing from the traditional treatment of solid geometry as a succession of formal propositions and proofs, this text aims to prepare the student for college work in mathematics and engineering. It is based on plane geometry and elementary algebra, and includes carefully stated assumptions, stimulating oral questions, and well-assorted written exercises. A distinctive feature is a simplified method of drawing three dimensional figures in orthographic perspective with a patented trimetric ruler supplied with the book.

## COLLEGE ALGEBRA

*By Frederick S. Nowlan, University of British Columbia. 401 pages, $3.00*

• A comprehensive text for college freshmen. Distinguished by a more detailed and careful review of elementary material than is usual, and by the comprehensiveness with which the subject matter is developed, the book is both mathematically sound and easily understandable.

## PLANE TRIGONOMETRY

*By E. Richard Heineman, Texas Technological College. 253 pages, $2.00. With tables, $2.50. Tables alone, $0.90*

• An especially clear treatment of the subject, giving definite instructions for proving identities, providing a new rule for characteristics, using true-false questions, and reducing the need for memory work. Particularly helpful for those with little mathematical background.

## COLLEGE ALGEBRA

*By A. Adrian Albert, The University of Chicago. 278 pages, $3.00*

• A rigorous approach to the subject, stressing the fundamental unity in algebra by knitting together the study of the number systems of elementary mathematics, polynomials and allied functions, algebraic identities, equations, and systems of equations. Thus algebra is presented as a unified whole, instead of a miscellaneous collection of isolated topics.

***Send for copies on approval***

**McGRAW-HILL BOOK COMPANY, Inc.**

**330 West 42nd Street** **New York 18, N.Y.**

## A New Revised Edition of

# ANALYTIC GEOMETRY

### By Roscoe Woods

The revised edition of this text maintains the successful plan, the clarity, and accuracy for which the first edition became so favorably known. Changes in exposition have been made, based on actual classroom experience. The exercises have been revised with reference to arrangement, grading, numerical data, and wording. Excellent and varied problems are presented offering the student opportunity for testing his understanding of the underlying principles of the subject. The exercises include new approaches or hints for problem solving, and additional theory. They occur at frequent intervals, presenting ample opportunity for emphasis on points that present any difficulty. *Published March 16, 1948.* $3.50

# COLLEGE ALGEBRA

### By Harding and Mullins

A special feature of this review of elementary algebra is the frequent use of graphical methods for explaining or illustrating the processes of algebra. A large number of review problems are included, arranged in sets of odd-numbered and even-numbered problems, so that they may be used alternately with different classes. Answers are provided in the back of the book for the even-numbered problems. *Second Edition.* $2.90

**THE MACMILLAN COMPANY • 60 Fifth Avenue • New York 11**

## RINEHART MATHEMATICAL TABLES

BY HAROLD LARSEN, *Albion College*

The result of an extensive survey among mathematicians and engineers, this book offers the most frequently used mathematical tables. Decisions on the inclusion of these tables were made on the basis of tabulated opinions. Among these are tables on weights and measures, physical and mathematical constants, algebraic and geometric formulas, logarithms, trigonometric functions, functions for angles in radians and radians to degrees, minutes and seconds, and computations in compound interest. To be published this summer.

*Prob. 265 pp., prob. $1.50*

## ELEMENTS OF FORMAL LOGIC

BY AMBROSE AND LAZEROWITZ, *Smith College*

An introduction to the field of formal logic, written with a lucid style in natural sequence. The book presents to the student an over-all picture of the subject, including advances made in the study by modern logicians. The authors have stressed a strong ground work in general propositions and an acquaintance with formal deductive systems. Essentials of Aristotelian logic are included and further incorporated within the framework of the extended, present-day subject. To be published this summer.

*Prob. 324 pp., prob. $3.50*

## COLLEGE ALGEBRA

BY REAGAN, OTT, AND SIGLEY

Thoroughly revised and reorganized, a new edition of College Algebra will be ready for publication in the Fall. College Algebra has been widely used and acclaimed for its unconventional inductive approach, for the inclusion of a large selection of problems, and for the use of review, interspersed throughout the text where needed.

*Prob. 512 pp., prob. $3.50*

# 5

# THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

PUBLIC LIBRARY
DETROIT

VOLUME 55 NUMBER 5

CONTENTS

The Force of Mortality Function . . . . . F. C. SMITH 277
Some Convexity Properties of Surfaces of Negative Curvature . . . . . E. F. BECKENBACH 285
Some Theorems on Cyclic Polygons Inscribed in a Circle . . . . . TAKIS LADOPOULOS 301
Mathematical Notes . . H. T. MCADAMS, BRYANT TUCKERMAN 308
Classroom Notes. . . . . J. D. MANCILL, A. V. BAEZ 311
Elementary Problems and Solutions . . . . . 317
Advanced Problems and Solutions . . . . . 320
Recent Publications . . . . . 323
Clubs and Allied Activities . . . . . 325
News and Notices . . . . . 327
The Mathematical Association of America . . . . . 332
October Meeting of the Indiana Section . . . . . 332
Calendar of Future Meetings . . . . . 334

MAY 1948